

Jutta Rath

On the Asymptotic Structure of Powers of Monomial Ideals

DISSERTATION

submitted in fulfilment of the requirements for the degree of

Doktorin der Technischen Wissenschaften



Alpen-Adria-Universität Klagenfurt

Fakultät für Technische Wissenschaften

Supervisors

**Priv.-Doz. Sen. Scientist Dr.
Roswitha Rissner**

University of Klagenfurt
Department of Mathematics

**Prof.
Clemens Heuberger**
University of Klagenfurt
Department of Mathematics

Expert Reviewers

**Assoc. Prof.
Carmelo Antonio Finocchiaro**
University of Catania
Department of Mathematics
and Computer Science

**Asst. Prof.
Lorenzo Guerrieri**
Jagiellonian University of Kraków
Department of Algebraic Geometry
and Number Theory

Klagenfurt, July 2025

AFFIDAVIT

I hereby declare in lieu of an oath that

- the submitted academic paper is entirely my own work and that no auxiliary materials have been used other than those indicated,
- I have fully disclosed all assistance received from third parties during the process of writing the thesis, including any significant advice from supervisors,
- any contents taken from the works of third parties or my own works that have been included either literally or in spirit have been appropriately marked and the respective source of the information has been clearly identified with precise bibliographical references (e.g. in footnotes),
- I have fully and truthfully declared the use of generative models (Artificial Intelligence, e.g. ChatGPT, Grammarly Go, Midjourney) including the product version,
- to date, I have not submitted this paper to an examining authority either in Austria or abroad and that
- when passing on copies of the academic thesis (e.g. in printed or digital form), I will ensure that each copy is fully consistent with the submitted digital version.

I am aware that a declaration contrary to the facts will have legal consequences.

Jutta Rath m.p.

Klagenfurt am Wörthersee, July 2025.

ABSTRACT

This thesis investigates the asymptotic behaviour of algebraic invariants associated with powers of monomial ideals, with particular emphasis on associated primes and the number of minimal generators.

After introducing the necessary background on monomial ideals and their combinatorial interpretations, we investigate the number of generators of high powers of bivariate monomial ideals. We establish an explicit bound beyond which the number of minimal generators becomes polynomial in the power, and provide a method for constructing the minimal generating sets of these powers from certain subideals of a fixed power, thereby reducing computational complexity. These results facilitate the effective computation of Hilbert functions and related invariants.

We then study the structure of Buchberger graphs and their relation to associated primes, introducing the lcm-complex to generalize results from strongly generic to arbitrary monomial ideals. This framework allows for a combinatorial characterization of associated primes in terms of simplicial complexes derived from least common multiples of the minimal generators of an ideal.

For monomial ideals in three variables, we apply the structural results from the bivariate case to derive bounds on the stability index of associated primes, considering cases based on the number and structure of minimal primes. Finally, we turn to monomial ideals in an arbitrary number of variables and address the problem of bounding the copersistence index—the power after which the sequence of associated primes of powers of an ideal is weakly decreasing. We present a method to derive bounds based on systems of linear inequalities that encode information about associated primes. Our approach yields upper bounds for the copersistence index that improve the existing bound by an exponential factor.

These results contribute to a better understanding of the asymptotic properties of monomial ideals, particularly the behaviour of their associated primes and minimal generators under powers, and provide new tools for their analysis within both algebraic and combinatorial frameworks.

ZUSAMMENFASSUNG

Diese Dissertation beschäftigt sich mit dem asymptotischen Verhalten von Potenzen von Monomialidealen. Im Mittelpunkt stehen dabei insbesondere die assoziierten Primideale, die Anzahl minimaler Erzeuger sowie die Potenz, ab der sich ein stabiles Verhalten einstellt und anfängliche Irregularitäten nicht mehr auftreten.

Zu Beginn werden die notwendigen Grundlagen zu Monomialidealen und deren Potenzen eingeführt. Anschließend analysieren wir die Struktur bivariater Monomialideale. Ein zentrales Ergebnis ist eine explizite Beschreibung hoher Potenzen solcher Ideale, basierend auf der Verknüpfung der sogenannten *Staircase-Diagramme* bestimmter Teilideale einer konkreten Potenz. Das ab einer gewissen Potenz auftretende periodische Muster in diesen Diagrammen ermöglicht eine explizite Beschreibung der minimalen Erzeuger aller höheren Potenzen und damit auch ihrer Anzahl. Diese Reduktion der strukturellen Analyse auf eine einzelne Potenz erlaubt eine wesentlich effizientere Berechnung der minimalen Erzeuger hoher Potenzen.

Im weiteren Verlauf untersuchen wir die Struktur von Buchberger-Graphen und deren Zusammenhang mit assoziierten Primidealen. Zur Verallgemeinerung bekannter Resultate von stark generischen auf beliebige Monomialideale wird der sogenannte *lcm-Komplex* eingeführt. Dieses Konzept erlaubt eine kombinatorische Charakterisierung der assoziierten Primideale in Form von Simplicialkomplexen, die sich aus den kleinsten gemeinsamen Vielfachen der minimalen Erzeuger ergeben.

Für Monomialideale in drei Variablen übertragen wir die strukturellen Erkenntnisse aus dem bivariaten Fall und leiten Schranken für den Stabilitätsindex der assoziierten Primideale ab. Dabei zeigt sich, dass sich Ideale in drei Variablen im Wesentlichen drei Fällen zuordnen lassen, die sich anhand ihrer minimalen Primideale unterscheiden.

Abschließend betrachten wir den sogenannten *Kopersistenzindex*—die Potenz, ab der die Folge der assoziierten Primideale schwach monoton fallend ist—für Monomialideale in beliebig vielen Variablen. Hierzu entwickeln wir eine Methode auf Basis linearer Ungleichungssysteme, die Informationen über die assoziierten Primideale kodieren. Unser Ansatz liefert obere Schranken, die unabhängig von der konkreten Wahl des Ungleichungssystems sind und bestehende Resultate um einen exponentiellen Faktor verbessern.

Insgesamt leisten die erzielten Ergebnisse einen Beitrag zum tieferen Verständnis der asymptotischen Eigenschaften von Monomialidealen—insbesondere im Hinblick auf das Verhalten ihrer assoziierten Primideale und minimalen Erzeuger bei Potenzbildung—und eröffnen neue Perspektiven für deren algebraische und kombinatorische Analyse.

ACKNOWLEDGEMENTS

First and foremost, I would like to express my deepest gratitude to my supervisor, Roswitha Rissner. I could not have asked for a more dedicated, supportive, and inspiring mentor. Thank you for sharing your knowledge, for always making time for discussions, for your invaluable advice, and for your constant encouragement. Our discussions were one of the most enjoyable aspects of this journey and greatly deepened my appreciation for research. I am very grateful to have had such a kind, reliable, and encouraging person as a supervisor. Thank you for all the time and effort you invested in me.

I am also sincerely grateful to my co-supervisor, Clemens Heuberger, whose influence on my academic path began long before my PhD—during my very first semester as a student. Your lectures left a lasting impression and your constant encouragement played a big role in my decision to pursue a doctorate. Throughout my PhD, your steady support, and insightful input have been invaluable. Thank you for being such a consistent and positive presence throughout this journey.

I would also like to express my gratitude to the Department of Mathematics at the University of Klagenfurt for all the opportunities I was offered and the support I received during this time. I want to thank all my colleagues at the department for making it such a welcoming, and joyful environment.

My doctoral studies in the doc.funds doctoral school “Modeling–Analysis–Optimization of discrete, continuous, and stochastic system” were generously funded by the Austrian Science Fund (FWF) [10.55776/DOC78]. I want to thank the FWF and the principal investigator, Michaela Szölgényi, for the funding, the opportunities, and the favourable environment. As part of the project, I had the opportunity to participate in great conferences, summer schools, and workshops.

I have benefitted greatly from research experience abroad, and for this I would like to thank the research council of the University of Klagenfurt, and my doc.funds doctoral program (FWF DOC 78) for their generous support. Additionally, I am grateful to my hosts and host institutions: Irem Portakal and the mathematics department of the Max Planck Institute for mathematics in the sciences, and Susan Morey and the mathematics department at Texas State University.

I would also like to express my gratitude to my examiners, Carmelo Antonio Finocchiaro and Lorenzo Guerrieri. Thank you for sharing your time and expertise in evaluating this thesis.

I would like to thank all my long-time and newly-found friends. Thank you, Nathan, for always being there, supporting me, and for bringing so much joy into my life. Thank you,

Diane, for adding so much to this period of my life, and for making our office such a warm and joyful place—you made me look forward to coming to work every day. Thank you, Tobi, for regularly brightening my days and for all the delicious food and shared laughter. Thank you, Daniel and Leanne, for keeping me grounded, for the fun activities, and the many insightful and enjoyable conversations. Thank you, Teresa, for being by my side in Klagenfurt from the very first day and for sharing this journey with me. Thank you, Moritz, for your kindness and your contagious enthusiasm for mathematics. Thank you, Laura, Sarah, and Patrick, for helping me maintain a healthy balance in life and for the friendship and support you offered along the way.

Zuletzt möchte ich ganz besonders meiner Familie für ihre bedingungslose Unterstützung danken. Ich danke meinen Eltern, Astrid und Michael, meiner Schwester Johanna sowie Lilith und Emil. Ich danke von Herzen meinem Bruder Benjamin, der mich zu diesem Studium—und zu so vielem mehr—ermutigt hat.

CONTENTS

Notation	xi
List of Figures	xiii
1 Introduction	1
2 Preliminaries	5
2.1 General facts about monomial ideals	5
2.1.1 Staircase diagrams	7
2.1.2 Primary decomposition and associated primes of monomial ideals	8
2.1.3 Associated primes and localization	12
2.1.4 Conditions for the maximal ideal to be associated	15
2.1.5 Common divisors of the minimal generators of an ideal	17
2.2 Monomial ideals and related objects	18
2.2.1 Edge ideals and cover ideals of graphs	19
2.2.2 The Newton polyhedron, integral closure, and reductions	21
2.3 Powers of monomial ideals	22
2.3.1 Minimal generators of powers of monomial ideals	27
2.3.2 Associated primes of powers of monomial ideals	28
3 Minimal generating sets of large powers of bivariate monomial ideals	31
3.1 Integral closures of bivariate monomial ideals	32
3.2 The role of persistent generators	34
3.3 Ideals with regular staircase factors	38
3.4 Minimal generating sets of powers	50
3.4.1 Runtime in practice	57
4 Buchberger graphs and the lcm-complex	59
4.1 The Buchberger graph of a monomial ideal	59
4.1.1 Definition and some properties of the Buchberger graph	59
4.1.2 Complete subgraphs of $\text{Buch}(I)$ and $\mathfrak{m}$	60
4.2 The lcm-complex of an ideal	62
4.2.1 Definition and properties of the lcm-complex	62
4.2.2 Connections between $\mathfrak{L}(I)$ and $\text{Ass}(R/I)$	64
5 Associated primes of powers of monomial ideals in three variables	67
5.1 Preliminaries	68
5.2 Upper bounds for the stability index	68

5.2.1	One minimal prime: $\text{Min}(R/I) = \{(x, y)\}$	70
5.2.2	Two minimal primes: $\text{Min}(R/I) = \{(x, y), (x, z)\}$	72
6	Bounds on the copersistent index of general monomial ideals	83
6.1	Graded factor modules related to systems of linear inequalities	84
6.1.1	Estimates on the generators of the solution spaces	87
6.1.2	Homogeneous elements of the factor module $\mathcal{H}_b/\mathcal{H}_0$	89
6.2	Upper bounds for the copersistence index $B_{\sup}^I$	90
6.2.1	Notes on the bound-parameters d , s , and r	90
6.2.2	Copersistence of prime ideals $\mathfrak{p}(M)$	90
6.2.3	Approach 1: $\mathfrak{m} \in \text{Ass}(R/I^n)$ if and only if $(\text{sat}(I^n) \cap I^{n-1}) \neq I^n$	92
6.2.4	Approach 2: $\mathfrak{m} \in \text{Ass}(R/I^n)$ if and only if $(I^n : \mathfrak{m}) \neq I^n$	92
6.2.5	Approach 3: $\mathfrak{m} \in \text{Ass}(R/I^n)$ if and only if $\text{sat}(I^n) \neq I^n$	95
6.3	Comparison of the different approaches	96
	Bibliography	97

NOTATION

$A \cup B$	Union of A and B
$A \uplus B$	Disjoint union of A and B
$A \cap B$	Intersection of A and B
$ A $	Cardinality of A
$A \setminus B$	Set difference $\{x \in A \mid x \notin B\}$
$\subseteq$	Subset
$\subsetneq$	Proper subset
$\mathbb{N}$	The natural numbers
$\mathbb{N}_0$	The natural numbers and 0
$\mathbb{Z}$	The integers
$[n]$	The set $\{1, \dots, n\}$
$\text{conv}(A)$	The convex hull of a set A
$\text{cone}(A)$	The cone generated by a set A
$\emptyset$	The empty set
$\mathbb{Q}$	The rational numbers
$\mathbb{R}$	The real numbers
$[a, b]$	Closed interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$
R	A commutative ring with unity
(F)	The ideal generated by a set F
k	Field of characteristic 0
$\sqrt{I}$	Radical of an ideal I
$I : J$	The ideal quotient of two ideals I and J
$\text{Ass}(R/I)$	Set of associated primes of an ideal I in the ring R
$\oplus$	Direct sum
$\ker \varphi$	The kernel of a map φ
$\bar{I}$	Integral closure of an ideal I
$\max$	Maximum
$\min$	Minimum
$\lceil x \rceil$	Ceiling function
lcm	Least common multiple
gcd	Greatest common divisor
K_r	Complete graph on r vertices
$\text{span}_k A$	The span over k of the elements of A

LIST OF FIGURES

2.1	An example of a staircase of an ideal in two variables.	7
2.2	An example of the staircase of an ideal in three variables.	8
2.3	Staircase diagrams of primary ideals.	9
2.4	A decomposition of a staircase diagram into primary components.	10
2.5	A different decomposition of a staircase diagram into primary components.	10
2.6	An example of how localisation can be used to compute the associated primes of an ideal.	14
2.7	A visualisation of how associated primes can be read off from the staircase diagram of an ideal.	15
2.8	The staircases of $I : \mathfrak{m}$ and $\text{sat}(I)$	16
2.9	The minimal vertex covers of a graph.	20
2.10	The Newton polyhedron of a monomial ideal in two variables	22
2.11	A comparison of the symbolic and ordinary power of a monomial ideal.	23
3.1	The behaviour of the integral closure of a monomial ideal under shifts.	32
3.2	Ideal with regular staircase factors.	38
3.3	The partition of an ideal with regular staircase factors based on y -degrees.	39
3.4	Visualisation of the sets L , M , and R in Example 3.3.5.	42
3.5	Visualisation of the link of ideals.	43
3.6	r -segments of an ideal with regular staircase factors.	45
3.7	A visualisation of Corollary 3.3.13.	46
3.8	The concatenation components C_i	47
3.9	Visualisation of the $(s, \bullet)$ -stable components	51
4.1	The Buchberger graph of a monomial ideal.	60
4.2	An ideal whose Buchberger graph is a triangle.	61
4.3	The Buchberger graph of an ideal can change when dividing the ideal by its gcd.	62
4.4	The lcm-complex of a monomial ideal.	64
5.1	An illustration of how to apply Proposition 5.2.15.	75
5.2	Some staircases of monomial ideals with three variables and two minimal primes of height two.	75
6.1	Depiction of the argument in the proof of Proposition 6.1.13.	89

1. INTRODUCTION

Monomial ideals play a central role in commutative algebra and due to their many combinatorial interpretations, they are one of the main objects in combinatorial commutative algebra. There are several properties of monomial ideals that make them so useful.

Performing algebraic operations is often far simpler on monomials than on arbitrary polynomials. Thus, certain invariants are more easily determined for monomial ideals. It is more efficient to analyze monomial ideals computationally than arbitrary ideals, as their structure allows for more efficient algorithmic techniques in computer algebra systems.

Monomial ideals are closely related to combinatorial objects such as graphs, simplicial complexes, and posets, which allows algebraic techniques to be applied to combinatorial problems and vice versa. The study of the connection between Algebra and Combinatorics was encouraged by Stanley's proof of the Upper Bound Conjecture [59] for simplicial spheres. This marked the beginning of a deep and ongoing interplay between algebra and combinatorics. For instance, properties of finite simple graphs can be studied through the squarefree monomial cover and edge ideals. This connection between graphs and monomial ideals has been first explored by Fröberg [17], Villarreal [66], and Simis, Vasconcelos, and Villarreal [57].

With the development of Gröbner basis theory, many problems concerning polynomials can be reduced to analogous problems on monomials by defining a suitable term order. This allows to deduce certain properties of an ideal from its initial monomial ideal. Bruno Buchberger introduced Gröbner bases in his 1965 Ph.D. thesis, named after his advisor Wolfgang Gröbner. His algorithm to compute Gröbner bases builds on what he called S -polynomials. Buchberger's criterion states that a set of polynomials F is a Gröbner basis if each S -polynomial can be reduced to zero. The S -polynomials depend on the initial terms of the polynomials in F and yield a set of elements that generate the module of first syzygies of the (monomial) initial ideal of F . This set is generally not a minimal generating set. According to Buchberger's second criterion, knowing a smaller one allows the algorithm to compute Gröbner bases more efficiently. Thus, the study of monomial ideals and their syzygies play a central role in understanding the structure of general ideals.

One of the main topics of this thesis, and a fundamental tool for gaining insight into the structure of an ideal, is the study of its associated primes. Broadly speaking, associated primes generalize the notion of prime factorization for integers and can be viewed as the minimal building blocks of an ideal. The concept often becomes clearer when demonstrated through examples involving structures related to ideals. For instance, edge ideals are monomial ideals generated by the edges of a graph; their associated primes correspond to the minimal vertex covers of the graph (cf. [6, Lemma 2.13]). The cover ideal is the

Alexander dual of the edge ideal, and its associated primes correspond to the edges of the graph (cf. [6, Lemma 2.12]). For finite simple hypergraphs, there is an intrinsic relation between the associated primes of powers of the cover ideal and colouring properties of the underlying graph [16]. In algebraic geometry, the associated primes of the defining ideal of a variety correspond to the defining ideals of its irreducible components (cf. [11, Section 3.8]). And in perhaps the simplest and most intuitive case, the associated primes of an ideal in the ring of integers correspond precisely to the prime factors of the integer that generates the ideal.

Main objectives. A key property of monomial ideals is that products are again monomial ideals. In particular, if I is a monomial ideal then all powers I^n are again monomial ideals. This naturally leads to the question of how certain invariants of a monomial ideal behave as we consider its powers. The aim of this thesis is to better understand the asymptotic structure of powers of monomial ideals. While many invariants behave highly irregular for small powers and are sometimes only understood in special cases, it is known that the changes in certain invariants eventually stabilise. For instance, the sequences $(\text{depth}(R/I^n))_{n \in \mathbb{N}}$ and $(\text{Ass}(R/I^n))_{n \in \mathbb{N}}$ are eventually constant [4, 5], the regularity of I^n is given by an affine function of n for all sufficiently large n , i.e., there exist constants a and b such that $\text{reg}(I^n) = an + b$ for $n \gg 0$ [10, 35], and there exists a polynomial function in n that describes the number of generators $\mu(I^n)$ for n large enough. We focus on the associated primes and number of generators, with the aim of bounding the power of stabilisation and describing the stabilised form.

Overview of the thesis.

Chapter 2: Preliminaries. We begin by recalling fundamental definitions and key properties of monomial ideals and their associated prime ideals. We use staircase diagrams as a tool to visualize certain properties. Especially, we point out their relation to the associated primes of an ideal. An overview of some related combinatorial objects is provided, including a selection of results that are relevant for this thesis. Finally, we discuss powers of monomial ideals, providing a brief summary of the tools and frameworks used in their study and introducing some questions explored in this context.

Chapter 3: Minimal generating sets of large powers of bivariate monomial ideals.

This chapter is based on the submitted preprint [48] and is joint work with Roswitha Rissner. We provide an explicit description of the minimal generating sets of large powers of bivariate monomial ideals. Specifically, we show that for sufficiently large $s \in \mathbb{N}$, every higher power $I^{s+\ell}$ can be constructed from certain subideals of I^s . We further show that such an s can be chosen to satisfy $s \leq \mu(I)(d^2 - 1) + 1$, where d is a constant determined by the degrees of the minimal generators of I , bounded above by the maximal x - or y -degree appearing in $G(I)$. This yields an explicit description of $G(I^{s+\ell})$ in terms of $G(I^s)$, which significantly reduces the computational complexity of determining high powers of bivariate monomial ideals. Further, this description enables the computation of the Hilbert polynomial of I , and thus $\mu(I^n)$ for all $n \geq s$. We include runtime measurements for

the SageMath implementation, which is available as an ancillary file on the arXiv page of [48].

Chapter 4: Buchberger graphs and the lcm-complex. *The content of this chapter is the object of a paper in preparation.* Buchberger graphs were originally introduced to reduce the generating set of the module of first syzygies of the initial ideal of a set of monomials to enable a faster computation of Gröbner bases. We explore some properties of Buchberger graphs of ideals in three variables and study the connections of Buchberger graphs of ideals in r variables to the associated primes of the underlying ideal. Since some results apply only to strongly generic ideals, we introduce the lcm-complex of an ideal as a tool to extend these findings to general monomial ideals. We provide a description of the associated primes of an ideal in terms of the faces of its lcm-complex.

Chapter 5: Stability of ideals in three variables. *The content of this chapter is the object of a paper in preparation jointly with Rowitha Rissner.* We build on the structural results for bivariate monomial ideals from Chapter 3 to establish bounds on the stability index of associated primes of monomial ideals in three variables. Specifically, when the ideal has one or two minimal prime ideals of height two, the stabilisation point of the staircases associated with certain related bivariate monomial ideals—described in Chapter 3—serves as a bound for the stability index of the ideal in three variables.

Chapter 6: Bounding the copersistence index in any number of variables. *This chapter is based on joint work with Clemens Heuberger and Roswitha Rissner [30], published in the Journal of Linear Algebra and its Applications.* Lê Tuấn Hoa [31] gave an upper bound for the stability index of arbitrary monomial ideals. This bound depends on the generators of the ideal and is obtained by separately bounding the powers of I after which the sequence of associated primes is non-decreasing and non-increasing, respectively. In this chapter, we focus on the latter and call the smallest such number the copersistence index. We take up the proof idea of Lê Tuấn Hoa, who exploits a certain system of inequalities whose solution sets store information about the associated primes of powers of I . However, these proofs are entangled with a specific choice for the system of inequalities. In contrast to that, we present a generic ansatz to obtain an upper bound for the copersistence index that is uncoupled from this choice of the system. We establish properties for a system of inequalities to be eligible for this approach to work. We construct two suitable inequality systems to demonstrate how this ansatz yields upper bounds for the copersistence index and compare them with Hoa's. One of the two systems leads to an improvement of the bound by an exponential factor.

2. PRELIMINARIES

We start with a short historical overview of ideals and their associated primes. By the Fundamental Theorem of Number Theory, every integer can be uniquely decomposed into a product of prime numbers. The prime numbers appearing in this factorization, along with their exponents, are uniquely determined. This property of uniquely decomposing elements into a product of elements that cannot be reduced further does not hold in any commutative ring with identity. For example, in the ring $\mathbb{Z}[\sqrt{-5}]$ there are two fundamentally different factorizations of the element 6, namely $2 \cdot 3 = 6 = (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5})$. As a workaround, Ernst Kummer and Richard Dedekind established in the 1800s that uniqueness is restored by factoring elements into products of ideals, which are, in a sense, idealized versions of numbers.

In the 1900s, Emanuel Lasker and Emmy Noether observed that decomposing ideals into finite intersections—rather than products—of irreducible ideals, i.e., ideals that cannot be written as a nontrivial intersection of two ideals, offers certain advantages. The Lasker-Noether Theorem (1905, 1921) states that every ideal in a commutative Noetherian ring has such a decomposition.

Interpreting this in terms of ideals, we see that any ideal in the ring $\mathbb{Z}$ can be uniquely expressed as an intersection of ideals generated by powers of prime numbers. For example, consider $140 = 2^2 \cdot 5 \cdot 7$. The ideal generated by 140, consisting of all its multiples, is given by the intersection of the ideals (2^2) , (5) , and (7) . That is, $(140) = (2^2) \cap (5) \cap (7)$ is the unique way to write (140) as an intersection of ideals generated by prime powers.

However, when generalizing such decompositions to other settings, uniqueness is generally lost. Despite this, the underlying prime ideals—the *associated primes*—remain uniquely determined.

2.1 General facts about monomial ideals

We introduce some terminology and facts about monomial ideals and associated primes of monomial ideals. For a thorough introduction we refer to Chapter 1 in Jürgen Herzog's and Takayuki Hibi's textbook [25] on monomial ideals.

An **ideal** in a commutative ring R is a nonempty subset that is closed under addition and invariant under multiplication by elements of R . We write $(g_1, \dots, g_n)$ for the ideal generated by the elements $g_1, \dots, g_n \in R$, that is, $(g_1, \dots, g_n) = \{r_1 g_1 + \dots + r_n g_n \mid r_1, \dots, r_n \in R\}$. Let $k[x_1, \dots, x_r]$ be the polynomial ring in r variables over a field k of characteristic 0. A **monomial** is a product of powers of the variables $x_1^{a_1} \cdots x_r^{a_r}$ for $a_1, \dots, a_r \in \mathbb{N}_0$.

Notation 2.1.1. For $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{N}_0^r$, we use the notation $x^{\mathbf{a}} := x_1^{a_1} \cdots x_r^{a_r}$.

Notation 2.1.2. Let $f = x^{\mathbf{a}} \in k[x_1, \dots, x_r]$ with $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{N}_0^r$. For every $i \in \{1, \dots, r\}$, we write $\deg_i f := a_i$ (sometimes also $\deg_{x_i} f$). Further, we write $\deg f := (a_1, \dots, a_r)$.

Naturally, the set of all monomials forms a k -basis of R . A **monomial ideal** is an ideal generated by monomials and again, the monomials contained in a monomial ideal form a k -basis of that ideal. Consequently, a monomial ideal is described entirely by its monomials. In fact, it suffices to choose those monomials which are minimal with respect to divisibility.

Proposition 2.1.3 (cf. [25, Proposition 1.1.6]). *Each monomial ideal I has a unique minimal set of monomial generators. More precisely, let G be the set of monomials in I which are minimal with respect to divisibility. Then G is the unique minimal set of monomial generators.*

Notation 2.1.4. For a monomial ideal I , we denote by $G(I)$ the set of its minimal monomial generators. Further, we denote by $\mu(I) := |G(I)|$ the number of minimal generators of I .

Monomial ideals behave nicely under algebraic operations, as listed below.

Fact 2.1.5 (cf. [25, Chapter 1.2]). *Let $I, J \subseteq R$ be monomial ideals. Then the following properties hold:*

- (1) *The sum $I + J$ is a monomial ideal and $G(I + J) \subseteq G(I) \cup G(J)$.*
- (2) *The intersection $I \cap J$ is a monomial ideal and $G(I \cap J) \subseteq \{\text{lcm}(u, v) \mid u \in G(I), v \in G(J)\}$.*
- (3) *The product $I \cdot J$ is a monomial ideal and $G(I \cdot J) \subseteq G(I) \cdot G(J)$.*
- (4) *The colon ideal $I : J := \{w \in R \mid wJ \subseteq I\}$ is a monomial ideal.*
- (5) *The radical $\sqrt{I} := \{w \in R \mid \exists n \in \mathbb{N} : w^n \in I\}$ is a monomial ideal.*

Throughout, we often implicitly use these basic properties of monomial ideals without referring to Fact 2.1.5.

Remark 2.1.6. If I is a monomial ideal, then the radical of I is a squarefree monomial ideal and it is generated by the monomials obtained by setting every non-zero exponent in the generators of I to 1.

2.1.1 Staircase diagrams

We often identify a monomial in r variables with the point in $\mathbb{R}^r$ defined by its exponent vector. We use this translation to visualise ideals in two or three variables. Moreover, the minimal generators of a monomial ideal in two variables x and y are always of the form $x^{a_1}y^{b_1}, \dots, x^{a_s}y^{b_s}$, where

$$a_1 < a_2 < \dots < a_s, \text{ and} \\ b_1 > b_2 > \dots > b_s.$$

Definition 2.1.7. We say a monomial u **properly divides** a monomial v , write $u \mid_p v$, if $u \mid v$ and the degree of u is different from the degree of v in every variable that occurs in v . We call a monomial $v \in I$ a **surface monomial** of I if it is not proper divisible by any monomial in I .

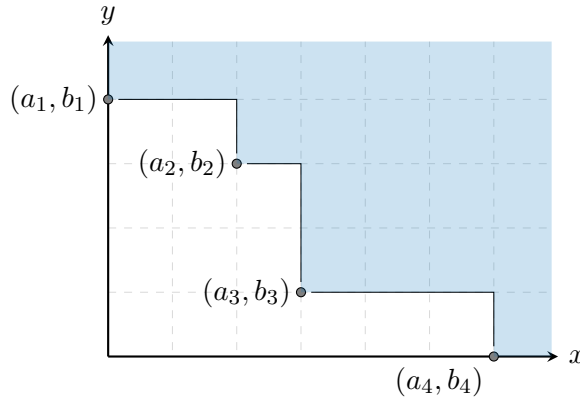


Fig. 2.1: A visualisation of the ideal (y^4, x^2y^3, x^3y, x^6) in two variables. All the grid points in the shaded area correspond to monomials in the ideal. The grid points on the staircase-line connecting the minimal generators correspond to the surface monomials of the ideal.

Definition 2.1.8 (cf. [43, Definition 3.6]). The **staircase surface** (or **staircase diagram**) of a monomial ideal I in r variables is the topological boundary of the space of vectors $v \in \mathbb{R}^r$ for which there is some monomial $f \in I$ satisfying $\deg_i f \leq v_i$ for all $1 \leq i \leq r$.

Remark 2.1.9. The integer points on the staircase surface of an ideal are precisely its surface monomials.

Example 2.1.10. Figure 2.2 illustrates the staircase surface of the monomial ideal

$$I = (x^2z, x^2y, yz, xy^3) \subseteq k[x, y, z].$$

All grid points on the coloured surface correspond to surface monomials of I . The empty boxes indicate that the surface continues infinitely in the corresponding directions.

The “inward-pointing corners” (white dots) correspond to the minimal generators of I . The “outward-pointing corners” are marked with black dots and we will refer to them as “outer corners”.

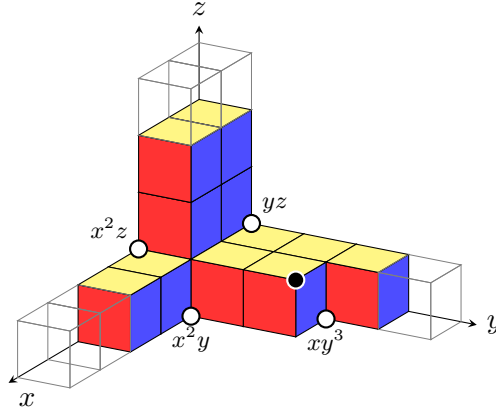


Fig. 2.2: The 3-dimensional staircase surface of the ideal $I = (x^2z, x^2y, yz, xy^3)$.

Definition 2.1.11. A surface monomial m is called **outer corner** if $x_1 \cdots x_r \mid m$ and

$$\frac{m}{\prod_{j \neq i} x_j}$$

is a surface monomial for every $i \in \{1, \dots, r\}$.

Throughout this thesis, we use staircase diagrams to help illustrate key results and examples, as they provide a powerful means of visualising certain properties of monomial ideals.

2.1.2 Primary decomposition and associated primes of monomial ideals

Primary decomposition is a standard topic in most introductory texts on commutative algebra; see, for example, Atiyah and Macdonald [2, Chapter 4]. In this section, while reviewing the fundamental definitions, we place particular emphasis on the behaviour and properties of *monomial ideals* in this context.

Definition 2.1.12. An ideal I in a ring R is called **primary** if for every product $a \cdot b \in I$, either $a \in I$ or $b^n \in I$ for some $n \in \mathbb{N}$. Denoting by $\mathfrak{p}$ the prime ideal $\sqrt{I}$, we then say that I is **$\mathfrak{p}$ -primary**.

Remark 2.1.13. Monomial prime and primary ideals can be characterized by their minimal generating sets.

- A monomial ideal $I \subseteq k[x_1, \dots, x_r]$ is primary if and only if for every $i \in \{1, \dots, r\}$, whenever x_i divides any minimal generator of I , then $x_i^n \in G(I)$ for some $n \in \mathbb{N}$, i.e., its staircase diagram is bounded in the direction of x_i (see Figure 2.3).
- A monomial ideal is prime if and only if it is minimally generated by a subset of the variables.

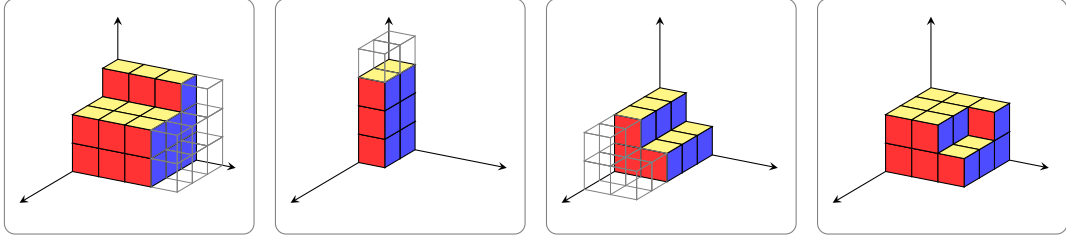


Fig. 2.3: From left to right: (x^3, xz^2, z^3) is (x, z) -primary; (x^2, y) is (x, y) -primary; (y^2, yz, z^2) is (y, z) -primary; (x^3, xy^2z, y^3, z^2) is (x, y, z) -primary.

Definition 2.1.14. Let I be an ideal in a ring R . A decomposition $I = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ for primary ideals $Q_1, Q_2, \dots, Q_n \subseteq R$ is called a **primary decomposition**. A primary decomposition is called **irredundant**, if

- $\sqrt{Q_1}, \dots, \sqrt{Q_n}$ are distinct, and
- for all $j \in \{1, \dots, n\}$, $I \neq \bigcap_{i \neq j} Q_i$.

Remark 2.1.15. By splitting up products of variables, we find that every monomial ideal has a primary decomposition with monomial primary components: If I has a minimal generator f that is not a power of a variable, then we can factor $f = f_1 \cdot f_2$, where $\gcd(f_1, f_2) = 1$, $f_1 \neq 1$, and $f_2 \neq 1$. Then

$$I = (J + (f_1)) \cap (J + (f_2)),$$

where J is the ideal with minimal generators $G(I) \setminus \{f\}$. Repeating this process until every generator of each component is a power of a variable leads to the desired decomposition. To obtain an irredundant decomposition, we combine all components that have the same radical: If $\mathfrak{p} = \sqrt{Q_1} = \sqrt{Q_2}$, then we replace Q_1 and Q_2 with the intersection $Q_1 \cap Q_2$. Note that $\sqrt{Q_1 \cap Q_2} = \mathfrak{p}$.

Example 2.1.16. To compute a primary decomposition of the monomial ideal given by $I = (x^3z, y^2z^2, x^2yz^2, y^4)$, we split up all appearing products step-by-step (Remark 2.1.15):

$$\begin{aligned}
 (x^3z, y^2z^2, x^2yz^2, y^4) &= (x^3, y^2z^2, x^2yz^2, y^4) \cap (z, y^2z^2, x^2yz^2, y^4) \\
 &= (x^3, y^2z^2, x^2yz^2, y^4) \cap (z, y^4) \\
 &= ((x^3, y^2, x^2yz^2, y^4) \cap (x^3, z^2, x^2yz^2, y^4)) \cap (z, y^4) \\
 &= (x^3, y^2, x^2yz^2) \cap (x^3, z^2, y^4) \cap (z, y^4) \\
 &= (x^3, y^2, x^2) \cap (x^3, y^2, y) \cap (x^3, y^2, z^2) \cap (x^3, z^2, y^4) \cap (z, y^4) \\
 &= (x^2, y^2) \cap (x^3, y) \cap (x^3, y^2, z^2) \cap (x^3, z^2, y^4) \cap (z, y^4).
 \end{aligned}$$

We can stop at this point, since the components in the last line are primary according to Remark 2.1.13. Using Remark 2.1.6, we compute their radicals to be (x, y) , (x, y, z) , and (y, z) . To obtain an irredundant primary decomposition, we combine the components that

have the same radical:

$$(x^2, y^2) \cap (x^3, y) = (x^3, x^2y, y^2), \quad (x^3, y^2, z^2) \cap (x^3, y^4, z^2) = (x^3, y^4, z^2).$$

We end up with the irredundant primary decomposition

$$I = (x^3, x^2y, y^2) \cap (x^3, y^4, z^2) \cap (z, y^4).$$

See Figure 2.4 for the decomposition of the staircase of I .

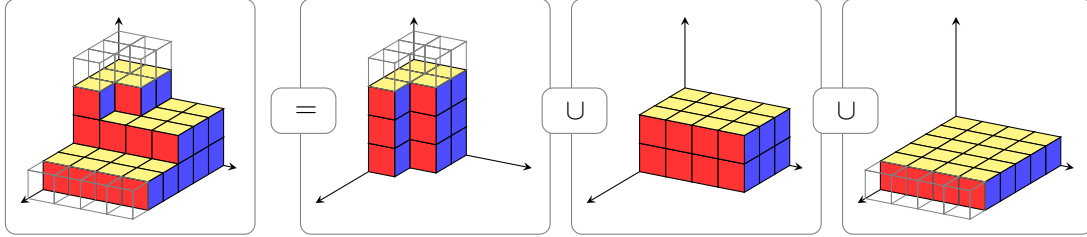


Fig. 2.4: The left-most staircase is the one of $I = (x^3z, y^2z^2, x^2yz^2, y^4)$. We decompose the staircase into a union of three staircases that correspond to the primary components of I .

Remark 2.1.17. We can modify the (x, y, z) -primary component of the ideal in Example 2.1.16 to obtain a different irredundant primary decomposition. For instance, it can be replaced with $(x^4, x^3y^2, x^3z, y^4, x^2z^2, y^2z^2, z^4)$. This alteration does not affect the overall intersection, as can be clearly seen from the staircase decomposition illustrated in Figure 2.5.

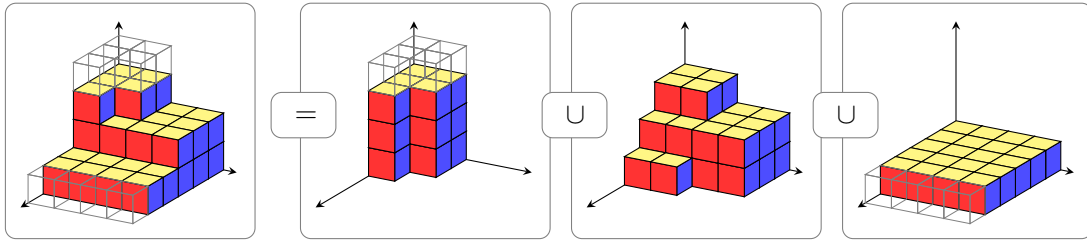


Fig. 2.5: A different primary decomposition of $I = (x^3z, y^2z^2, x^2yz^2, y^4)$.

Definition 2.1.18. Let I be an ideal in a Noetherian ring R with irredundant primary decomposition $I = Q_1 \cap \cdots \cap Q_n$. Then the elements of the set

$$\text{Ass}(R/I) := \{\sqrt{Q_1}, \dots, \sqrt{Q_n}\}$$

are called **associated primes** of I . The associated primes which are minimal with respect to inclusion are called **minimal primes** of I and are denoted by $\text{Min}(R/I)$. Every associated prime that is not minimal is called **embedded prime**.

We state a second definition of associated primes that does not require the use of primary

decomposition and applies to arbitrary (not necessarily Noetherian) rings. In the Noetherian case, this definition agrees with the one above (cf. [25, Chapter 1.3.2]).

Definition 2.1.19. For any ideal I in a ring R , a prime ideal $\mathfrak{p}$ is an **associated prime** of I if there exists an element $w \in R$ such that

$$\mathfrak{p} = I : w = \{r \in R \mid rw \in I\}.$$

The element w is called a **witness** of $\mathfrak{p}$ with respect to I .

Example 2.1.20. In Example 2.1.16 we computed an irredundant primary decomposition of the ideal $I = (x^3z, y^2z^2, x^2yz^2, y^4)$. From this decomposition, we obtain the set of associated primes:

$$\text{Ass}(R/I) = \{(x, y), (y, z), (x, y, z)\}.$$

Its minimal primes of I are (x, y) and (y, z) , while (x, y, z) is an embedded prime. According to Definition 2.1.19, these associated primes can also be identified via witnesses as follows:

$$(x, y) = I : x^2z^2, \quad (y, z) = I : x^3y^3, \quad (x, y, z) = I : x^2y^3z.$$

For monomial ideals, the set of associated primes can be described as follows:

Fact 2.1.21 (cf. [25, Corollary 1.3.10]). *If I is a monomial ideal in $R = k[x_1, \dots, x_r]$, then*

$$\text{Ass}(R/I) = \{\mathfrak{p} \text{ prime ideal} \mid \text{there exists } x^a \in R \text{ such that } \mathfrak{p} = I : x^a\},$$

that is, monomial witnesses always exist.

Notation 2.1.22. For $r \in \mathbb{N}$ we write $[r] := \{1, \dots, r\}$.

Notation 2.1.23. Let M be a subset of $[r]$. We denote by $\mathfrak{p}(M) := (x_i \mid i \in M) \subseteq R$, the prime ideal generated by the variables x_i , $i \in M$. Further, we write $\mathfrak{m}_R := (x_1, \dots, x_r)$ for the unique maximal monomial ideal in R . We simply write $\mathfrak{m}$ if the ring is clear from the context.

Notation 2.1.24. For a monomial ideal I in $k[x_1, \dots, x_r]$, we denote by

$$\text{supp}(I) := \{i \in [r] \mid x_i \text{ divides a minimal generator of } I\}$$

and we call this set the **support** of I .

Fact 2.1.25 (cf. [25, Section 1.3]). *Let I be a monomial ideal in $k[x_1, \dots, x_r]$. Then*

$$\text{Ass}(R/I) \subseteq \{\mathfrak{p}(M) \mid M \subseteq \text{supp}(I)\}.$$

Remark 2.1.26. As an immediate consequence of Fact 2.1.25, we get that if $x_i \nmid g$ for all $g \in G(I)$, then $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}(R/I)$.

2.1.3 Associated primes and localization

Localizing at a suitable multiplicative set can simplify the computation of the associated primes of a monomial ideal.

Notation 2.1.27. Let $M \subseteq [r]$. We denote by

$$R_M := (k[x_i \mid i \notin M] \setminus \{0\})^{-1} R$$

the localization of R at $k[x_i \mid i \notin M] \setminus \{0\}$. If I is an ideal in R , then we write $I_M := IR_M$.

Remark 2.1.28. In contrast to the localization of R at $\mathfrak{p}(M)$, the ring R_M remains a polynomial ring. Specifically, it is a polynomial ring in $|M|$ variables and over an extended field, that is, $R_M = k'[x_i \mid i \in M]$ where $k' = k(x_i \mid i \notin M)$.

Fact 2.1.29 (cf. [11, Theorem 3.1]). *Associated primes of ideals behave well with respect to localization, that is,*

$$\text{Ass}(R_M/I_M) = \left\{ \mathfrak{p}R_M \mid \mathfrak{p} \in \text{Ass}(R/I) \text{ and } x_i \notin \mathfrak{p} \text{ for all } i \in [r] \setminus M \right\}.$$

Remark 2.1.30. Let $M \subseteq [r]$. By Fact 2.1.29 we have

$$\mathfrak{p}(M) \in \text{Ass}(R/I) \iff \mathfrak{m}_{R_M} = \mathfrak{p}(M)R_M \in \text{Ass}(R_M/I_M).$$

This equivalence allows us to focus on the maximal monomial ideal $\mathfrak{m}_R$ only. For non-maximal prime ideals $\mathfrak{p}(M)$ we localize to R_M where $\mathfrak{p}(M)R_M$ is maximal. To sum up, the following holds:

$$\text{Ass}(R/I) = \bigcup_{M \subseteq [r]} \left\{ \mathfrak{p}(M) \mid \mathfrak{m}_{R_M} \in \text{Ass}(R_M/I_M) \right\}.$$

In Example 2.1.16 and Remark 2.1.17, we observed that primary decompositions are generally not unique. However, for primary components corresponding to minimal primes, the following holds:

Fact 2.1.31 (cf. [40, Theorem 6.8(iii)]). *The primary components corresponding to minimal primes are uniquely determined by I . Specifically, if $\mathfrak{p} \in \text{Min}(R/I)$ then the $\mathfrak{p}$ -primary*

component of I is $\pi^{-1}(I_{\mathfrak{p}})$, where $\pi: R \rightarrow R_{\mathfrak{p}}$ is the canonical projection.

In particular, note that for $\mathfrak{p} = \mathfrak{p}(M)$, the projection π is equal to $\hat{\pi} \circ \varphi: R \rightarrow R_M \rightarrow R_{\mathfrak{p}}$ where $\hat{\pi}$ and φ are the canonical projections. Hence, if $\mathfrak{p}(M)$ is a minimal prime of I then $\varphi^{-1}(I_M) = \pi^{-1}(I_{\mathfrak{p}(M)})$ is the $\mathfrak{p}(M)$ -primary component of I .

Remark 2.1.32. Note that for any $M \subseteq [r]$, the preimage $\varphi^{-1}(I_M)$ of I_M under the map $\varphi: R \rightarrow R_M$ is equal to the saturation of I at $\mathfrak{p}([r] \setminus M)$, i.e.,

$$\varphi^{-1}(I_M) = I : \mathfrak{p}([r] \setminus M)^{\infty} = \bigcap_{i \in [r] \setminus M} I : x_i^{\infty}.$$

Remark 2.1.33. By Remark 2.1.30, the ideal (x_i) is associated to I if and only if $I_{\{i\}} \neq R_{\{i\}}$; that is, if and only if $x_i \mid g$ holds for all $g \in G(I)$. Furthermore, by Fact 2.1.31, it follows that in this case, $\varphi^{-1}(I_{\{i\}})$ is the uniquely determined (x_i) -primary component of I .

We illustrate the results of this section on an example:

Example 2.1.34. Let $I = (x^4w, x^3yz^2, x^2y^3wz, yz^2w, z^3w)$ be an ideal in $R = k[x, y, z, w]$. In this example, we adapt the notation from Notation 2.1.27 by replacing the set M with the set of the corresponding variables. We begin by localizing at $k[x] \setminus \{0\}$ which gives the ring $R_{\{y,z,w\}} = k'[y, z, w]$ and $I_{\{y,z,w\}} = (w, yz^2)$. (Note that the base field changes from k to k' , but this does not affect our arguments about associated primes.) We observe that $\mathfrak{m}_{R_{\{y,z,w\}}} = (y, z, w) \notin \text{Ass}(R_{\{y,z,w\}}/I_{\{y,z,w\}})$, and hence by Remark 2.1.30 it follows that

$$(y, z, w) \notin \text{Ass}(R/I).$$

Next, we localize further at $k'[y] \setminus \{0\}$, obtaining $R_{\{z,w\}} = k''[z, w]$ and $I_{\{z,w\}} := (z^2, w)$. Now, the maximal ideal (z, w) in this localized ring is associated, so

$$(z, w) \in \text{Ass}(R/I).$$

This process of localization allows us to compute the associated primes of I by progressively reducing the number of variables, simplifying the computations. Applying this technique, we obtain (see Figure 2.6):

$$\text{Ass}(R/I) \setminus \{(x, y, z, w)\} = \{(x, z), (x, w), (y, w), (z, w), (x, y, z)\}.$$

The remaining question is whether the maximal ideal (x, y, z, w) is also an associated prime of I . Using Figure 2.6 and Fact 2.1.31, we can directly identify the primary components of I corresponding to its minimal primes

$$\text{Min}(R/I) = \{(x, z), (x, w), (y, w), (z, w)\}.$$

For example, the (x, z) -primary components of I is

$$\varphi^{-1}(I_{\{x,z\}}) = (I : y^\infty) \cap (I : w^\infty) = (x^4, x^2z, z^2).$$

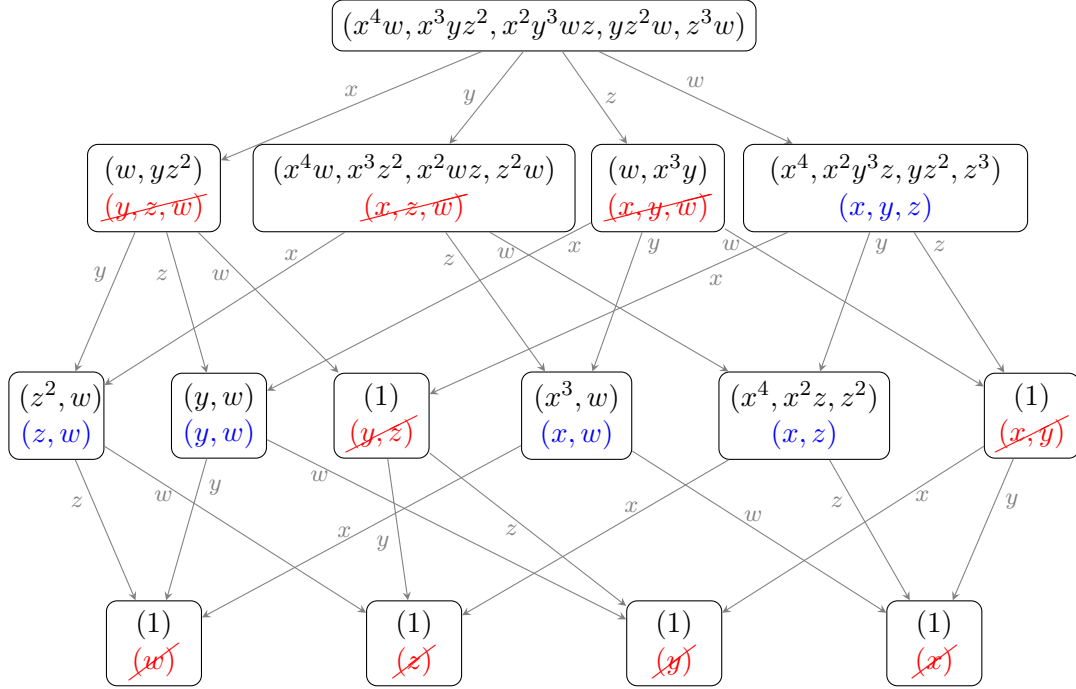


Fig. 2.6: We start on the top with the ideal I from Example 2.1.34. From top to bottom, in each layer we reduce the number of variables by one. We then check if the maximal ideal in the corresponding ring is associated (crossed out in red if it is not associated; blue if it is associated).

We conclude with a remark that combines the results of this section with the use of the staircase diagram as a visual tool for extracting information about $\text{Ass}(R/I)$:

Remark 2.1.35. Let I be a monomial ideal in $k[x, y, z]$.

- (1) The ideal (z) is associated to I if and only if no generator of I lies in the x - y -plane—that is, if every minimal generator has positive z -degree. This follows from Remark 2.1.33. Analogous statements hold for the primes (y) and (x) .
- (2) If the ideal (x, y) is associated to I , then the staircase diagram of I is unbounded in z -direction, that is, there exists no minimal generator g of I lying on the z -axis. To see this, we assume that $(x, y) = I : w$ for some monomial w . Then $z^n w \notin I$ must hold for all $n \in \mathbb{N}$, which is only possible if no power of z is in I .

The reverse implication of this statement holds under the assumption that neither (x) nor (y) is an associated prime of I . In that case, by (1), there exist minimal generators of the form $y^b z^d$ and $x^a z^c$. The exponents of these generators fulfill

$a \neq 0$ and $b \neq 0$ since otherwise, the staircase diagram of I would be bounded in z -direction. Thus, $I : z^\infty$ is (x, y) -primary and in particular, by Fact 2.1.31 and Remark 2.1.32 it is the (x, y) -primary component of I .

Again, analogous statements hold for (x, z) and (y, z) .

We provide examples in Figure 2.7.

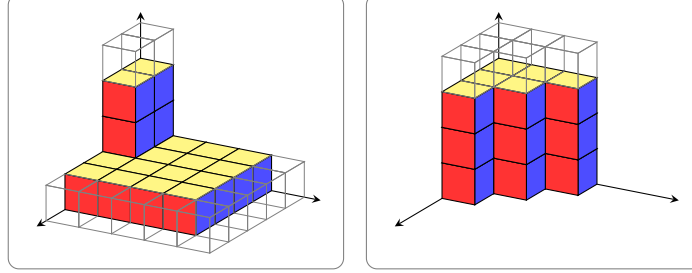


Fig. 2.7: The staircase **on the left** is of the ideal (x^2z, yz) . Every minimal generator of this ideal is divisible by z , hence (z) is associated. Further, (x) and (y) are not associated, thus, the fact that the staircase is unbounded in z -direction implies that (x, y) is associated. The staircase **on the right** belongs to the ideal (x^3, x^2y, xy^2, y^3) . It is unbounded in z -direction and has no minimal primes of height one. Therefore, (x, y) is associated to this ideal.

2.1.4 Conditions for the maximal ideal to be associated

By Remark 2.1.30, we have $\mathfrak{p}(M) \in \text{Ass}(R/I)$ if and only if $\mathfrak{m}_{R_M} \in \text{Ass}(R_M/I_M)$. This subsection therefore focuses on maximal ideals.

We start with the following well-known characterizing statement:

Fact 2.1.36. *Let I be a monomial ideal. Then $\mathfrak{m} \in \text{Ass}(R/I)$ if and only if $I : \mathfrak{m} \neq I$.*

Remark 2.1.37. There is a one-to-one correspondence between the witnesses of $\mathfrak{m}$ and the monomials in $I : \mathfrak{m} \setminus I$. Clearly, if w is a witness of $\mathfrak{m}$, then $w \in I : \mathfrak{m} \setminus I$. On the other hand, if $f \in I : \mathfrak{m} \setminus I$, then $f \cdot x_i \in I$ for all $i \in [r]$ and $f \notin I$, thus, $I : f = \mathfrak{m}$.

Corollary 2.1.38. *Let $I \subseteq k[x_1, \dots, x_r]$ be a monomial ideal. Then $\mathfrak{m} \in \text{Ass}(R/I)$ if and only if I has an outer corner. In particular, there is a one-to-one correspondence between the witnesses of $\mathfrak{m}$ and the outer corners of I .*

Proof. Let m be an outer corner of I . We claim that $w := m/x_1 \cdots x_r$ is a witness of $\mathfrak{m}$. First, observe that for each $i \in [r]$ the product $x_i \cdot w$ is a surface monomial by the definition of outer corners, and therefore lies in I . Moreover, if $f \mid w$, then $f \mid w \cdot x_1 \cdots x_r = m$ which is by definition a surface monomial and thus not properly divisible by f ; hence, there exists an $i \in [r]$ such that $\deg_i f = \deg_i m = \deg_i w + 1$, a contradiction. This shows that $w \notin I$ and therefore w is indeed a witness of $\mathfrak{m}$.

On the other hand, if w is a witness of $\mathfrak{m}$, then we show that $m := w \cdot x_1 \cdots x_r$ is an outer corner of I . Since $w \notin I$ but $w x_i \in I$ for every $i \in [r]$, it follows that

$$w \cdot x_i = \frac{m}{\prod_{j \neq i} x_j}$$

is a surface monomial. □

Combining the above, we observe that the staircase of $I : \mathfrak{m}$ is obtained from the staircase of I by removing all cubes whose endpoints correspond to an outer corner, see Figure 2.8.

Definition 2.1.39 (cf. [11, Section 15.10.6]). For a monomial ideal I , let

$$\text{sat}(I) := I : \mathfrak{m}^\infty = \bigcup_{k \in \mathbb{N}_0} (I : \mathfrak{m}^k)$$

be the **saturation** of I with respect to $\mathfrak{m}$.

Remark 2.1.40. Since $\text{sat}(I) = \bigcap_{i=1}^r (I : x_i^\infty)$ holds (cf. [36, Lemma 3.5.12]), $\text{sat}(I)$ is again a monomial ideal.

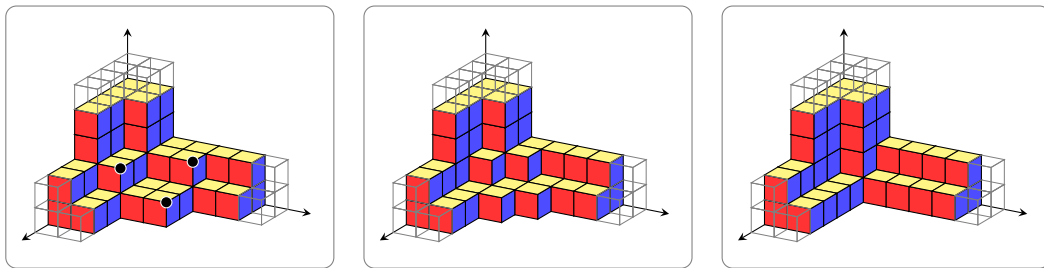


Fig. 2.8: Let $I = (x^4 z^2, x^4 y z, x^4 y^2, x^2 y z^2, x^2 y^2 z, y^2 z^2, x y^4 z, x^2 y^4)$. The staircase of I is **on the left**. Its three outer corners $x^4 y^2 z^2$, $x^4 y^4 z$, and $x^2 y^4 z^2$ are marked by black dots. The staircase of $I : \mathfrak{m}$ is obtained by removing the cubes corresponding to the outer corners. The result is illustrated **in the middle**. By repeating this process we end up with the staircase **on the right**, which belongs to $\text{sat}(I)$.

Fact 2.1.41 ([9, Chapter 4, Exercise 14]). Let I be a monomial ideal. Then $\mathfrak{m}$ is associated to I if and only if $\text{sat}(I) \neq I$ holds.

Lemma 2.1.42. Let I be a monomial ideal. For any $n \in \mathbb{N}$, $\text{sat}(I^n) \neq I^n$ if and only if $\text{sat}(I^n) \cap I^{n-1} \neq I^n$.

Proof. Note that the following inclusions hold:

$$I^n \subseteq \text{sat}(I^n) \cap I^{n-1} \subseteq \text{sat}(I^n).$$

Therefore, $\text{sat}(I^n) \cap I^{n-1} \neq I^n$ implies $\text{sat}(I^n) \neq I^n$. For the reverse implication, note

that $I^n = I^n : \mathfrak{m}$ implies $I^n = \text{sat}(I^n)$. Hence, if $I^n \neq \text{sat}(I^n)$ then there exists a monomial $x^a \in (I^n : \mathfrak{m}) \setminus I^n$. Let $i \in [r]$. Since $x^a x_i \in I^n$, there exist $g_1, \dots, g_n \in I$ such that

$$x^a x_i = g_1 \cdots g_n.$$

This further implies that $x_i \mid g_j$ for some j , say $j = 1$, and hence

$$x^a = \frac{g_1}{x_i} g_2 \cdots g_n \in I^{n-1}.$$

Since $I^n : \mathfrak{m} \subseteq \text{sat}(I^n)$, we conclude that $x^a \in (\text{sat}(I^n) \cap I^{n-1}) \setminus I^n$. □

Remark 2.1.43. To summarize this subsection, we provide a list of statements that characterize when the maximal ideal $\mathfrak{m}$ is associated to I^n . The following statements are equivalent:

- (1) $\mathfrak{m} \in \text{Ass}(R/I^n)$,
- (2) $I^n : \mathfrak{m} \neq I^n$ (Fact 2.1.36),
- (3) $\text{sat}(I^n) \neq I^n$ (Fact 2.1.41),
- (4) $\text{sat}(I^n) \cap I^{n-1} \neq I^n$ (Lemma 2.1.42).

2.1.5 Common divisors of the minimal generators of an ideal

Many properties of monomial ideals—such as the number of minimal generators—remain invariant under shifts. It is often convenient to shift ideals prior to performing computations. We develop several relevant properties below.

Lemma 2.1.44. *Let I be a monomial ideal in $k[x_1, \dots, x_r]$ and m a monomial. Then*

$$(I : m)^n \subseteq I^n : m^n$$

holds for all $n \in \mathbb{N}$. If $m \mid g$ for all $g \in G(I)$, then equality $(I : m)^n = I^n : m^n$ holds.

Proof. Let $f = f_1 \cdots f_n$ be in $(I : m)^n$, for $f_j \in I : m$. Then $f_1 \cdots f_n \cdot m^n = f_1 m \cdots f_n m \in I^n$. If m divides every minimal generator of I , then

$$(I : m)^n = \left(\frac{1}{m} I \right)^n = \frac{1}{m^n} I^n = I^n : m^n.$$

□

Notation 2.1.45. For a monomial ideal I , we denote by $\text{gcd}(I)$ the greatest common divisor of all monomials in I .

Definition 2.1.46. For a monomial ideal I , we write $I_{\mathfrak{z}} := I : \gcd(I)$. We call the ideal I **anchored**, if $I_{\mathfrak{z}} = I$.

Remark 2.1.47. By Lemma 2.1.44, $(I^n)_{\mathfrak{z}} = (I_{\mathfrak{z}})^n$ holds for all $n \in \mathbb{N}$.

Clearly, $\mu(I) = \mu(I_{\mathfrak{z}})$. We now show that common divisors of the generators of I do not play a role for the associated primes of I^n .

Proposition 2.1.48. Let I be a monomial ideal in $r > 1$ variables and let $\mathbf{t} \in \mathbb{N}_0^r$ such that $x^{\mathbf{t}}$ divides all of the generators of I . Then for all $n \in \mathbb{N}$, $\mathfrak{m} \in \text{Ass}(R/I^n)$ if and only if $\mathfrak{m} \in \text{Ass}(R/(I : x^{\mathbf{t}})^n)$.

Proof. By Lemma 2.1.44 it suffices to show the assertion for $n = 1$. If $\mathfrak{m}$ is associated to $I : x^{\mathbf{t}}$, then there exists a $x^{\mathbf{w}}$ such that

$$\mathfrak{m} = (I : x^{\mathbf{t}}) : x^{\mathbf{w}} = I : x^{\mathbf{t}} x^{\mathbf{w}} = I : x^{\mathbf{t} + \mathbf{w}}$$

holds and therefore $\mathfrak{m} \in \text{Ass}(R/I)$.

Conversely, let $\mathfrak{m} \in \text{Ass}(R/I)$ with witness $x^{\mathbf{w}}$. Since all generators of I are divisible by $x^{\mathbf{t}}$ and $x^{\mathbf{w}} x_i \in I$, we have $\mathbf{t} \leq \mathbf{w} + \mathbf{e}_i$ for all $i \in [r]$, where $\mathbf{e}_i$ is the i -th unit vector. Hence, $\mathbf{t} \leq \mathbf{w}$ and $x^{\mathbf{w}} = x^{\tilde{\mathbf{w}}} x^{\mathbf{t}}$ for some $x^{\tilde{\mathbf{w}}}$. Therefore,

$$\mathfrak{m} = I : x^{\mathbf{w}} = I : x^{\mathbf{t}} x^{\tilde{\mathbf{w}}} = (I : x^{\mathbf{t}}) : x^{\tilde{\mathbf{w}}},$$

i.e., $\mathfrak{m} \in \text{Ass}(R/(I : x^{\mathbf{t}}))$. □

Corollary 2.1.49. Let I be a monomial ideal and $x^{\mathbf{t}}$ be a divisor of all the generators of I . Then

$$\text{Ass}(R/I^n) \setminus \{(x_1), \dots, (x_r)\} = \text{Ass}(R/(I : x^{\mathbf{t}})^n) \setminus \{(x_1), \dots, (x_r)\}.$$

Proof. Let $M \subseteq [r]$ with $|M| > 1$. We apply Proposition 2.1.48 to the maximal monomial ideal $\mathfrak{m}_{R_M}$ in the localization R_M of R . Remark 2.1.30 then yields that $\mathfrak{p}(M)$ is associated to I^n if and only if $\mathfrak{p}(M) \in \text{Ass}(R/(I : x^{\mathbf{t}})^n)$. □

2.2 Monomial ideals and related objects

We introduce some combinatorial objects that often appear in the context of monomial ideals. Broadly speaking, by associating an ideal to a given object and then analysing the ideal, we can extract valuable information about the original object. For instance, the associated primes of certain ideals corresponding to graphs can reveal insights into the graph's colouring properties. Conversely, this relationship also allows us to apply combinatorial methods to derive results about monomial ideals.

2.2.1 Edge ideals and cover ideals of graphs

We introduce edge ideals and cover ideals of finite simple hypergraphs. These ideals provide well-studied examples of the interplay between combinatorics and commutative algebra, wherein graph-theoretic results are employed to derive properties of monomial ideals, and vice versa. Edge ideals were first introduced for finite simple graphs by Villarreal [66] and extended to hypergraphs by Hà and Van Tuyl [21].

Edge and cover ideals have attracted a great deal of interest and have been heavily studied in the last decades. We refer to [44], and [65] for a great overview, while we only scratch the surface here and focus on the associated primes of these ideals.

Definition 2.2.1. A **finite simple hypergraph** is a pair $\mathcal{H} = (V_{\mathcal{H}}, E_{\mathcal{H}})$ where $V_{\mathcal{H}} = \{x_1, \dots, x_r\}$, the set of **vertices** of $\mathcal{H}$, and $E_{\mathcal{H}} = \{E_1, \dots, E_s\}$, where the E_i are subsets of $V_{\mathcal{H}}$ of cardinality at least two, and $E_i \not\subseteq E_j$ for $i \neq j$, the **edges** of $\mathcal{H}$.

By the condition that every edge has at least two elements, $\mathcal{H}$ has no loops. The condition that no two edges are contained in each other ensures that there are no multiple edges. If every edge has exactly two elements, then we call $\mathcal{H}$ a **finite simple graph** which we usually denote by G .

Definition 2.2.2. Let $\mathcal{H} = (\{x_1, \dots, x_r\}, \{E_1, \dots, E_s\})$ be a finite simple hypergraph. The **edge ideal** of $\mathcal{H}$ is the ideal

$$I_{\mathcal{H}} := \left(\prod_{x_i \in E} x_i \mid E \in E_{\mathcal{H}} \right) \subseteq k[x_1, \dots, x_r].$$

Remark 2.2.3. The assignment $\mathcal{H} \mapsto I_{\mathcal{H}}$ gives a natural one-to-one correspondence between hypergraphs and squarefree monomial ideals (ignoring isolated vertices).

Definition 2.2.4. A **vertex cover** of a finite simple hypergraph $\mathcal{H}$ is a subset of the vertices $W \subseteq V_{\mathcal{H}}$ such that $W \cap E \neq \emptyset$ holds for all $E \in E_{\mathcal{H}}$. A vertex cover is called **minimal** if no proper subset is also a vertex cover.

Definition 2.2.5. Let $\mathcal{H} = (\{x_1, \dots, x_r\}, \{E_1, \dots, E_s\})$ be a finite simple hypergraph. The **cover ideal** of $\mathcal{H}$ is the ideal

$$J_{\mathcal{H}} := \left(\prod_{x_i \in W} x_i \mid W \text{ is a minimal vertex cover of } \mathcal{H} \right) \subseteq k[x_1, \dots, x_r].$$

Remark 2.2.6. For a hypergraph $\mathcal{H}$, the two ideals $I_{\mathcal{H}}$ and $J_{\mathcal{H}}$ are squarefree monomial ideals in $k[x_1, \dots, x_r]$. Thus, by Remark 2.1.15, a decomposition into prime ideals always exists. Further, they are Alexander duals of each other, that is, if $I_{\mathcal{H}}$ has primary

decomposition

$$I_{\mathcal{H}} = (x_i \mid i \in M_1) \cap \cdots \cap (x_i \mid i \in M_\ell),$$

for some $M_1, \dots, M_\ell \subseteq \{1, \dots, r\}$, then

$$J_{\mathcal{H}} = \left(\prod_{i \in M_j} x_i \mid j \in \{1, \dots, \ell\} \right).$$

For the generalized Alexander duality for arbitrary monomial ideals, we refer to Chapter 5 of the textbook by Miller and Sturmfels [43]. In particular, the associated primes of the edge and cover ideal of a hypergraph are given by

$$\begin{aligned} \text{Ass}(R/I_{\mathcal{H}}) &= \{(x_{i_1}, \dots, x_{i_k}) \mid \{x_{i_1}, \dots, x_{i_k}\} \text{ is a minimal vertex cover}\}, \\ \text{Ass}(R/J_{\mathcal{H}}) &= \{(x_{i_1}, \dots, x_{i_k}) \mid \{x_{i_1}, \dots, x_{i_k}\} \in E_{\mathcal{H}}\}. \end{aligned}$$

Example 2.2.7. We consider the finite simple graph

$$G = (\{x_1, \dots, x_4\}, \{\{x_1, x_3\}, \{x_1, x_4\}, \{x_1, x_2\}, \{x_3, x_4\}\}).$$

Then the edge ideal of G is $I_G = (x_1x_2, x_1x_3, x_1x_4, x_3x_4)$ and has primary decomposition

$$I_G = (x_1, x_3) \cap (x_1, x_4) \cap (x_2, x_3, x_4).$$

All minimal vertex covers of G are given by $\{x_1, x_3\}$, $\{x_1, x_4\}$, $\{x_2, x_3, x_4\}$, see Figure 2.9, hence the cover ideal of G is $J_G = (x_1x_3, x_1x_4, x_2x_3x_4)$.

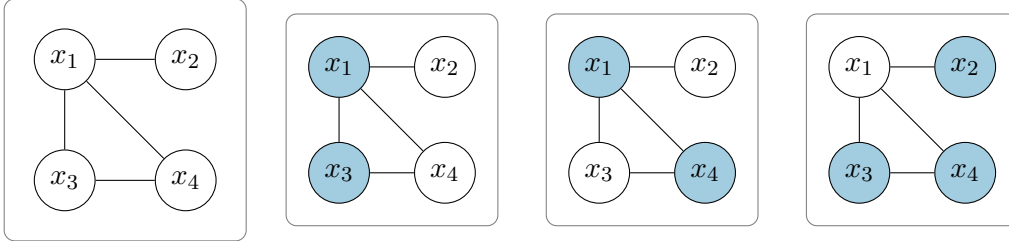


Fig. 2.9: On the left is the graph G from Example 2.2.7. The other three figure illustrate the three minimal vertex covers of G .

We recall some terminology about colourings of graphs:

Definition 2.2.8. A k -colouring of a hypergraph $\mathcal{H}$ is a partition of $V_{\mathcal{H}} = C_1 \uplus \cdots \uplus C_k$ into k disjoint sets such that for every $e \in E_{\mathcal{H}}$, we have $e \not\subseteq C_i$ for all $1 \leq i \leq k$. The **chromatic number** $\chi(\mathcal{H})$ of $\mathcal{H}$ is the minimal k , such that $\mathcal{H}$ has a k -colouring. Further, $\mathcal{H}$ is called k -**chromatic** if $\chi(\mathcal{H}) = k$.

Remark 2.2.9. A graph G admits a k -colouring if there exists an assignment of k colours to its vertices such that no two adjacent vertices share the same colour.

Definition 2.2.10. A hypergraph $\mathcal{H}$ is called **critically k -chromatic** if $\mathcal{H}$ is k -chromatic but for every $v \in V_{\mathcal{H}}$, we have that the hypergraph obtained from $\mathcal{H}$ by deleting v from its vertices and removing all edges containing v , is ℓ -chromatic for some $\ell < k$.

Example 2.2.11. ■ The complete graph K_n , i.e., the graph with n vertices and an edge between each two of the vertices, is critically n -chromatic, as removing any vertex results in a graph with chromatic number equal to $n - 1$.

- If G is an odd cycle, i.e., $V_G = \{x_1, \dots, x_n\}$ for some odd $n \geq 3$, and

$$E_G = \{\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_n, x_1\}\},$$

then $\chi(G) = 3$. However, by removing any vertex, we obtain a path, which has chromatic number equal to two.

Fact 2.2.12 ([16, Corollary 4.6]). *Let $\mathcal{H}$ be a finite simple hypergraph and $P \subseteq V$ such that the induced hypergraph $\mathcal{H}_P$, i.e., the hypergraph with vertex set P and edges $\{E \in E_{\mathcal{H}} \mid E \subseteq P\}$, is critically k -chromatic. Then*

- (1) $P \in \text{Ass}(R/J_{\mathcal{H}}^k)$, and
- (2) $P \notin \text{Ass}(R/J_{\mathcal{H}}^{\ell})$ for any $1 \leq \ell < k$.

2.2.2 The Newton polyhedron, integral closure, and reductions

Definition 2.2.13. An element $f \in k[x_1, \dots, x_r]$ is **integral** over an ideal I , if there exist $k \in \mathbb{N}$ and for $1 \leq i \leq k$ an element $c_i \in I^i$ such that

$$f^k + c_1 f^{k-1} + \dots + c_{k-1} f + c_k = 0.$$

The set of all elements that are integral over I is called the **integral closure** of I and is denoted by $\bar{I}$. An ideal is called **integrally closed** if $I = \bar{I}$ and **normal** if all powers of I are integrally closed.

The integral closure of an ideal is again an ideal [32, Corollary 1.3.1] and if I is a monomial ideal, then it can be described as follows:

Fact 2.2.14 (cf. [25, Theorem 1.4.2]). *Let I be a monomial ideal in $k[x_1, \dots, x_r]$. Then its integral closure $\bar{I}$ is generated by all monomials f such that there exists an $n \in \mathbb{N}$ with $f^n \in I^n$.*

Definition 2.2.15. Let I be a monomial ideal in $k[x_1, \dots, x_r]$. The set $\text{conv}\{\mathbf{a} \mid x^{\mathbf{a}} \in I\} \subseteq \mathbb{R}^r$ is called the **Newton polyhedron** of I , denoted by $\mathcal{C}(I)$.

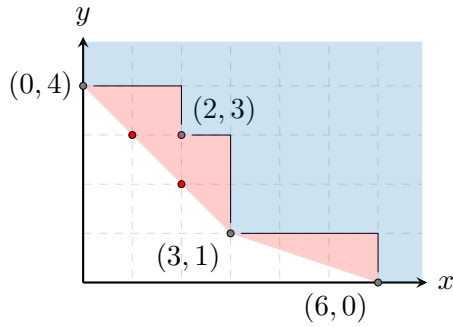


Fig. 2.10: The ideal (y^4, x^2y^3, x^3y, x^6) is not integrally closed as there are two grid points, corresponding to the monomials xy^3 and x^2y^2 , that are in $\mathcal{C}(I)$ but not in I .

Fact 2.2.16 (cf. [25, Corollary 1.4.3.]). *The integral closure $\bar{I}$ of a monomial ideal I is generated by the monomials x^a with $a \in \mathcal{C}(I)$.*

Remark 2.2.17. By Fact 2.2.16 an ideal I is integrally closed if and only if there are no integer points in the Newton polyhedron of I that do not correspond to monomials in I .

Definition 2.2.18. Given monomial ideals $I, J \subseteq k[x_1, \dots, x_r]$, the ideal J is said to be a (monomial) **reduction** of I if there exists some integer $m \geq 0$ such that $J I^m = I^{m+1}$. The least integer m for which this equation is fulfilled is called the **reduction number of I with respect to J** and the smallest reduction number amongst all reductions J of I is called the **reduction number of I** . A reduction J is called **minimal** monomial reduction if every monomial ideal L which is properly contained in J is not a reduction of I .

Remark 2.2.19. If J is a reduction of I and m is greater than or equal to the reduction number of I with respect to J , then

$$I^{m+\ell} = I^m J^\ell \text{ holds for all } \ell \geq 0.$$

Singla [58] determined the unique minimal monomial reduction of a monomial ideal I using its Newton polyhedron:

Fact 2.2.20 ([58, Proposition 2.1]). *Let I be a monomial ideal in $k[x_1, \dots, x_r]$ and let $\{a_1\}, \dots, \{a_\ell\}$ be the 0-dimensional faces of $\mathcal{C}(I)$. Then $J = (x^{a_1}, \dots, x^{a_\ell})$ is the unique minimal monomial reduction of I .*

2.3 Powers of monomial ideals

Powers of monomial ideals have been studied in many different contexts. After the pioneering work of Brodmann, proving that the associated primes of powers of an ideal $\text{Ass}(R/I^n)$ eventually become independent of n , and that the depth function $\text{depth}(R/I^n)$ is constant for n large enough [5, 4], a lot of research in that direction followed. For a great overview of recent developments in the research of powers of monomial ideals and their asymptotic behaviour, we refer to Carlini, Hà, Harbourne, and Van Tuyl's lecture notes [6].

Definition 2.3.1. Let I be an ideal and let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be the minimal primes of I . The n -th **symbolic power** of I is defined to be the ideal

$$I^{(n)} = Q_1 \cap \dots \cap Q_k,$$

where Q_i is the primary component of I^n corresponding to $\mathfrak{p}_i$.

Remark 2.3.2. It follows from the definition that the ordinary power of an ideal is always contained in its symbolic power, i.e., $I^n \subseteq I^{(n)}$ holds for all $n \in \mathbb{N}$. The question of when equality holds is known as the *Containment Problem*.

Fact 2.3.3 ([26, Lemma 3.1]). *If I is a monomial and $Q_1, \dots, Q_k$ are the primary components corresponding to the minimal primes of I , then for every $n \in \mathbb{N}$,*

$$I^{(n)} = Q_1^n \cap \dots \cap Q_k^n.$$

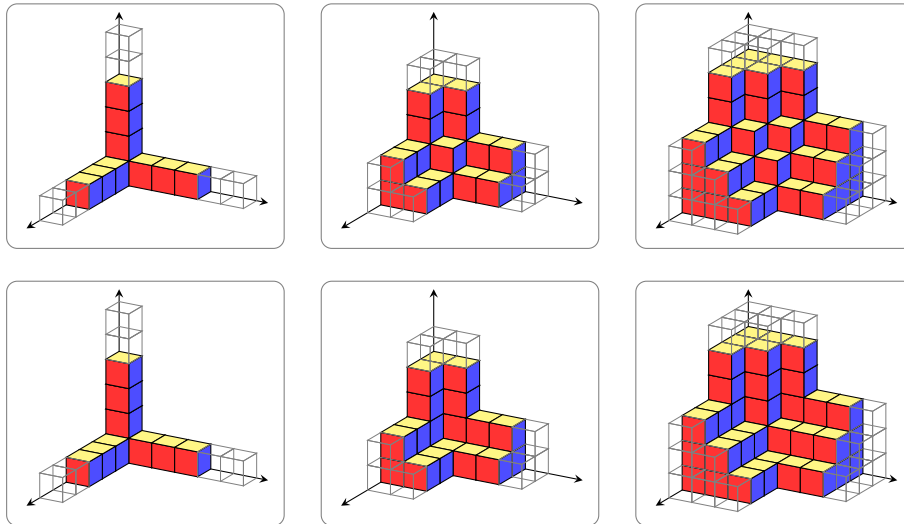


Fig. 2.11: The **left-most** staircase in the first row is of the ideal $I = (xy, xz, yz)$. Then the ordinary powers I^2 and I^3 follow. In the **second row** are from left to right the staircases of the symbolic powers $I^{(1)}$, $I^{(2)}$ and $I^{(3)}$.

Example 2.3.4. Let $I = (xy, xz, yz) \subseteq k[x, y, z]$. We compute that $\text{Min}(R/I) = \text{Ass}(R/I) = \{(x, y), (x, z), (y, z)\}$ and $I = (x, y) \cap (x, z) \cap (y, z)$. Thus, I is equal to its first symbolic power $I^{(1)}$. However,

$$I^2 = (x, y)^2 \cap (x, z)^2 \cap (y, z)^2 \cap (x^2, y^2, z^2) = I^{(2)} \cap (x^2, y^2, z^2),$$

see Figure 2.11.

Remark 2.3.5. Let G be a graph and I_G its edge ideal. Sullivant [61, Corollary 3.12] established how to obtain the second symbolic power of I_G in terms of edges and triangles

in G . Further, he gave a characterization of perfect graphs in terms of the symbolic powers of edge ideals, see [61, Theorem 3.10].

There are several algebraic structures that frequently appear in the study of powers of monomial ideals. We recall three fundamental ones: the Rees algebra, the associated graded ring, and the fibre ring of a monomial ideal. These structures are closely related and provide useful tools for organizing information, carrying out computations, and understanding the properties of ideal powers. To study powers of a monomial ideal, it is crucial to understand the relations among its minimal generators.

Definition 2.3.6. A **syzygy** or **linear relation** of $g_1, \dots, g_s \in \mathbf{k}[x_1, \dots, x_r]$ is a tuple $(f_1, \dots, f_s) \in \mathbf{k}[x_1, \dots, x_r]^s$ such that

$$f_1 g_1 + f_2 g_2 + \dots + f_s g_s = 0.$$

Definition 2.3.7. The **Rees algebra** of a monomial ideal I is defined as

$$\mathcal{R}(I) := \bigoplus_{n \geq 0} I^n t^n \subseteq R[t],$$

where t is a new variable.

If the minimal generators of I are $\{g_1, \dots, g_s\}$ then $\mathcal{R}(I) = R[g_1 t, \dots, g_s t]$. There is a natural homogeneous epimorphism of R -algebras

$$\Phi: R[w_1, \dots, w_s] \rightarrow \mathcal{R}(I),$$

where $R[w_1, \dots, w_s]$ is the polynomial ring over R in variables $w_1, \dots, w_s$, defined by

$$\Phi(w_i) = g_i t \quad \text{for } i = 1, \dots, s.$$

Thus, $\mathcal{R}(I) \simeq R[w_1, \dots, w_s] / \ker \Phi$. The kernel of Φ is called the defining ideal of $\mathcal{R}(I)$. Often, the defining ideal is considered to study Rees algebras. It contains the relations obtained from the syzygies of $g_1, \dots, g_s$, however, determining all defining relations is a difficult task in general.

Example 2.3.8. We consider the ideal $I = (x^2, xy, y^2)$ in $R = \mathbf{k}[x, y]$. The Rees algebra of I is $R[x^2 t, xy t, y^2 t] \subseteq \mathbf{k}[x, y][t]$. We compute all syzygies of the generators of I :

$$\begin{aligned} x(xy) - y(x^2) &= 0, \\ y(xy) - x(y^2) &= 0. \end{aligned}$$

Further, we obtain the relation $(xy)^2 - (x^2)(y^2) = 0$. In this case, these are all the defining

relations and we get the isomorphism

$$\mathcal{R}(I) \simeq \frac{R[w_1, w_2, w_3]}{(xw_2 - yw_1, yw_2 - xw_3, w_2^2 - w_1w_3)}.$$

Definition 2.3.9. The **associated graded ring** of I is

$$\text{gr}(I) := \bigoplus_{n \geq 0} I^n / I^{n+1} \simeq \mathcal{R}(I) / I\mathcal{R}(I),$$

where the multiplication of two homogeneous elements $a + I^{n+1} \in I^n / I^{n+1}$ and $b + I^{m+1} \in I^m / I^{m+1}$ is

$$(a + I^{n+1})(b + I^{m+1}) = ab + I^{m+n+1} \in I^{m+n} / I^{m+n+1}.$$

There is a strong relation between the associated graded ring of an ideal and the associated primes of the power of that ideal. Let $\mathfrak{m}\mathcal{R}(I)$ be the ideal of $\mathcal{R}(I)$ generated by $\mathfrak{m}$. The 0-th local cohomology module

$$H_{\mathfrak{m}\mathcal{R}(I)}^0(\text{gr}(I)) = \{g \in \text{gr}(I) \mid (\mathfrak{m}\mathcal{R}(I))^n g = 0 \text{ for some } n \in \mathbb{N}\}$$

can be written as a direct sum (cf. [10, Lemma 2.1])

$$H_{\mathfrak{m}\mathcal{R}(I)}^0(\text{gr}(I)) = \bigoplus_{n \geq 0} H_{\mathfrak{m}}^0(I^n / I^{n+1}).$$

For monomial ideals, the equality $\text{Ass}(I^n / I^{n+1}) = \text{Ass}(R / I^{n+1})$ holds for all $n \geq 0$, cf. [6, Lemma 2.5]. Combined with the above, this yields:

Fact 2.3.10. *The maximal ideal $\mathfrak{m}$ is associated to I^{n+1} if and only if $H_{\mathfrak{m}\mathcal{R}(I)}^0(\text{gr}(I))_n \neq 0$.*

In Chapter 6 we give an upper bound for the degrees of the homogeneous generators of $H_{\mathfrak{m}\mathcal{R}(I)}^0(\text{gr}(I))$ which we then use to study the stability of the associated primes of the powers of I .

Fact 2.3.11 ([56, Proposition 2.4]). *Let $\mathcal{R}(I)_+$ be the positive part of the $\mathcal{R}(I)$. Then*

$$\text{Ass}(R / I^n) \subseteq \text{Ass}(R / I^{n+1})$$

holds for all $n > \sup \{n \mid H_{\mathcal{R}(I)_+}^0(\text{gr}(I))_n \neq 0\}$.

Again, there is a useful description of the homogenous components, this time using the

Ratliff-Rush closure of an ideal, which is defined as

$$\tilde{I} = \bigcup_{m \geq 1} I^{m+1} : I^m.$$

For all $n > 0$ we then have $H_{\mathcal{R}(I)_+}^0(\text{gr}(I))_{n-1} \simeq (\tilde{I}^n \cap I^{n-1})/I^n$, cf. [31, Lemma 3.6].

Intuitively, $\tilde{I}$ consists of all elements that “behave like” elements of I at high powers—after multiplying by a high enough power of I , you land inside the next power. The Ratliff-Rush closure was first introduced by Ratliff and Rush [50], where they proved that I is a reduction of $\tilde{I}$ and $\tilde{I}^n = I^n$ holds for all $n \gg 0$.

The last structure that we want to introduce in this section is the fibre ring (or fibre cone) of an ideal. It was initially introduced in the context of blowup algebras. Geometrically, the fibre ring corresponds to the fibre of the blowup at the closed point corresponding to $\mathfrak{m}$.

Definition 2.3.12. The $\mathbf{k}$ -algebra $\mathcal{F}(I) = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ is called the **fibre ring** and its Krull dimension the **analytic spread** of I , denoted by $\ell(I)$.

The fibre ring captures the asymptotic growth of the minimal generators of I and its Hilbert function $H(\mathcal{F}(I), k) = \dim_{\mathbf{k}} I^k/\mathfrak{m}I^k$ counts the number of minimal generators of the powers of I . Also its dimension $\ell(I)$ is a fundamental invariant that provides insight into the long-term behaviour of the ideal.

Fact 2.3.13 ([32, Corollary 8.2.5]). *Let J be a reduction of I . Then the minimum number of generators of J is at least the analytic spread of I .*

Fact 2.3.14 ([28, Corollary 3.5]). *If $I \subseteq \mathbf{k}[x_1, \dots, x_r]$ is a so-called *polymatroidal ideal* (the exponents of the minimal generators represent the basis of a discrete polymatroid, cf. [28] for a definition), then $\ell(I) = r - \lim_{k \rightarrow \infty} \text{depth}(R/I^k)$.*

Remark 2.3.15. Let $G(I) = \{g_1, \dots, g_s\}$ and $\varphi: \mathbf{k}[t_1, \dots, t_s] \rightarrow \mathcal{F}(I)$ be the epimorphism between the polynomial ring $\mathbf{k}[t_1, \dots, t_s]$ and $\mathcal{F}(I)$ defined by $\varphi(t_i) = g_i + \mathfrak{m}I$. The generators of each homogeneous component $I^k/\mathfrak{m}I^k$ of the fibre ring correspond to the minimal generators $G(I^k)$. As all elements of $G(I^k)$ are products of elements of $G(I)$, the fibre ring is generated in degree one over the residue field $\mathcal{F}(I)_0 = R/\mathfrak{m}$. Thus, φ is surjective and

$$\mathcal{F}(I) \simeq \mathbf{k}[t_1, \dots, t_s]/\ker \varphi.$$

The ideal $\ker \varphi$ is called the **defining ideal** of the fibre ring.

In general, finding the defining ideal of $\mathcal{F}(I)$ can be challenging. In [27] and [29], the authors determined the defining ideals of the fibre rings of special classes of bivariate

monomial ideals.

2.3.1 Minimal generators of powers of monomial ideals

Given the minimal generators of a monomial ideal, the natural question whether we can determine the minimal generators of its powers, arises. Specifically, let I be a monomial ideal minimally generated by $g_1, \dots, g_s$. Then I^n is generated by all n -fold products of these generators, that is, by monomials of the form

$$\{g_1^{\alpha_1} \cdots g_s^{\alpha_s} \mid \alpha_i \in \mathbb{N}_0, \alpha_1 + \cdots + \alpha_s = n\}.$$

To determine $G(I^n)$, we must identify those monomials among these elements that are minimal with respect to divisibility.

As noted in Remark 2.3.15, the fibre ring $\mathcal{F}(I)$ of a monomial ideal I —and in particular its defining ideal—contain key information about the minimal generators of I . Although determining the defining ideal is generally a difficult task, the Hilbert function $H(\mathcal{F}(I), n) = \dim_{\mathbb{k}} I^n / \mathfrak{m} I^n$ provides a more accessible invariant: it counts the number of minimal generators of I^n and eventually agrees with a polynomial in n (cf. [25, Theorem 6.1.3]).

As a result, much of the existing literature concentrates on understanding the behaviour of $\mu(I^n)$, rather than characterizing the generators themselves. For any monomial ideal I and any $n \in \mathbb{N}$, the inequality

$$\mu(I^n) \leq n \cdot \mu(I)$$

always holds. Equality, however, occurs only in exceptional cases—namely, when there are no relations among the minimal generators of I .

Contrary to what one might expect, Eliahou, Herzog, and Saem [13] provide examples of bivariate monomial ideals that do not satisfy the inequality $\mu(I^2) \geq \mu(I)$:

Fact 2.3.16 ([13, Theorem 1.1]). *For every integer $m \geq 5$, there exists a monomial ideal I in $\mathbb{k}[x, y]$ such that $\mu(I) = m$ and $\mu(I^2) = 9$.*

More examples of monomial ideals whose minimal generators exhibit unexpected behavior in low powers can be found in [1, 19].

While the unexpected behaviour of small powers is fascinating, our attention in Chapter 3 shifts to the behavior of minimal generators in *large* powers of bivariate monomial ideals. The asymptotic properties of the Hilbert function suggest that, beyond a certain point, the structure of $G(I^n)$ stabilizes in a predictable way. In particular, we show that cancellations among the n -fold products of the generators of I eventually follow a regular pattern. Indeed, we provide an explicit description of the sets $G(I^n)$ of minimal generators of I^n for all n larger than a certain threshold.

2.3.2 Associated primes of powers of monomial ideals

While the prime factors of an integer $n \in \mathbb{Z}$ are the same as the prime factors of any power n^k of n , this statement cannot be expanded to associated primes in general. We begin this section with an example to motivate the following question:

Question 2.3.17. Given an ideal I , how does the set $\text{Ass}(R/I^n)$ change as n increases?

Example 2.3.18. $I = (xy, xz, yz) \subseteq k[x, y, z]$. Then

$$\text{Ass}(R/I) = \text{Min}(R/I) = \{(x, y), (x, z), (y, z)\}$$

and computations suggest that for all $n \geq 2$

$$\text{Ass}(R/I^n) = \text{Min}(R/I) \cup \{(x, y, z)\}.$$

The following fact gives a partial answer to Question 2.3.17, namely that the minimal primes remain unchanged when taking powers.

Fact 2.3.19. Let I be a monomial ideal and $n \in \mathbb{N}$. Then $\text{Min}(R/I) = \text{Min}(R/I^n)$.

Hence, the question reduces to understanding how the embedded primes of an ideal evolve. The behaviour of the sequence $(\text{Ass}(R/I^n))_{n \in \mathbb{N}}$ as n varies has been studied over the past few decades for various classes of ideals. In what follows, we introduce the necessary terminology and provide a brief overview of some classes of ideals whose associated primes have been the subject of such investigations.

Definition 2.3.20. An ideal I is called **normally torsion-free** if $\text{Ass}(R/I^k) \subseteq \text{Ass}(R/I)$ holds for all $k \in \mathbb{N}$.

If I is an ideal such that $\text{Ass}(R/I) = \text{Min}(R/I)$, i.e., I has no embedded primes, then I is normally torsion-free if and only if every power of I equals its symbolic power, cf. [67, Proposition 3.3.26].

Definition 2.3.21. An ideal I is said to have the **persistence property** if $\text{Ass}(R/I^n) \subseteq \text{Ass}(R/I^{n+1})$ holds for all $n \in \mathbb{N}$. A prime ideal $\mathfrak{p}$ is called **persistent** if $\mathfrak{p} \in \text{Ass}(R/I^n)$ implies that $\mathfrak{p} \in \text{Ass}(R/I^{n+1})$.

Examples of classes of ideals that satisfy the persistence property include edge ideals of simple undirected graphs [39, Theorem 2.15], cover ideals of perfect graphs [16, Corollary 5.11], and ideals whose powers are all integrally closed [49, Theorem 2.4].

However, the persistence property does not hold in general, not even for squarefree mono-

mial ideals [33, Theorem 11]. In fact, Weinstein and Swanson [69, Theorem 3.9] constructed families of monomial ideals whose sets of associated primes decrease with n . There are also known examples where $(\text{Ass}(R/I^n))_{n \in \mathbb{N}}$ is not even monotonic (cf. [33]). For general monomial ideals, little is known about how $\text{Ass}(R/I^n)$ changes with n .

Despite that, the asymptotic behaviour of the sequence of associated primes of powers of an ideal is well understood: In 1979, Brodmann gave an answer for Question 2.3.17 for $n \gg 0$.

Theorem 2.3.22 ([5]). *For sufficiently large n , the set $\text{Ass}(R/I^n)$ is independent of n .*

Definition 2.3.23. The smallest integer $\text{stab}(I)$ such that $\text{Ass}(R/I^n) = \text{Ass}(R/I^{\text{stab}(I)})$ for all $n \geq \text{stab}(I)$ is called the **stability index** of I . The set $\text{Ass}(R/I^{\text{stab}(I)})$ is called the **stable set** of I , and is sometimes also denoted by $\text{Ass}(R/I^\infty)$ or $\text{Ass}^\infty(R/I)$.

Not only does the set of associated primes of powers of an ideal stabilise, but further, Brodmann [4] also proved that the depth function of an ideal stabilises, i.e., for $n \gg 0$ then the function $\text{depth}(R/I^n)$ is constant. The smallest such n after which the depth function of an ideal I is constant is called the *index of depth stability* of I , denoted by $\text{dstab}(I)$. While neither $\text{stab}(I)$ is an upper bound for $\text{dstab}(I)$ nor the other way around, those two invariants are connected in a way by the relation $\text{stab}(I) \leq \max_{M \subseteq \text{supp}(I)} \text{dstab}(I_M)$ ([28, Proposition 2.1(c)]).

We now list a few special classes of ideals for which the stability index, or an explicit bound on it, is known: If I is a transversal polymatroidal ideal, i.e., a product of monomial prime ideals, then $\text{stab}(I) = 1$ ([28, Corollary 4.6]). Let G be a connected, non-bipartite graph with r vertices, σ vertices of degree one, and the smallest odd cycle of G has length $2k + 1$, then $\text{stab}(I) \leq r - k - \sigma$ ([8, Corollary 4.3]). Let J be the cover ideal of a finite simple hypergraph $\mathcal{H}$. Then the stability index of I is at least the chromatic number of $\mathcal{H}$ ([16, Corollary 4.9]).

Despite numerous results concerning specific classes of monomial ideals, relatively little is known about the stability index of general monomial ideals. In 2006, Hoa [31] provided a bound on the stability index applicable to arbitrary monomial ideals:

Fact 2.3.24 ([31, Theorem 3.1]). *Let I be a monomial ideal in $k[x_1, \dots, x_r]$ with $\mu(I) = s$ and let d be the maximal total degree appearing in the minimal generators of I . Then*

$$\text{stab}(I) \leq \max \left\{ d(rs + s + d)(\sqrt{r})^{r+1}(\sqrt{2}d)^{(r+1)(s-1)}, s(s+r)^4 s^{r+2} d^2 (2d^2)^{s^2-s+1} \right\}.$$

In Chapter 6, we discuss this bound in more detail and present refinements. In Chapter 5, we bound the stability index of monomial ideals in three variables.

3. MINIMAL GENERATING SETS OF LARGE POWERS OF BIVARIATE MONOMIAL IDEALS¹

When studying powers of a monomial ideal, its minimal generators play a crucial role. However, it is far from trivial to determine which of the n -fold products of (minimal) generators of I are minimal generators of I^n and very little is known—even for monomial ideals and even in the bivariate case.

The existing research mostly focuses on the number $\mu(I^n)$ of minimal generators of I^n rather than the actual set of minimal generators. Indeed, the emphasis lies on small powers, as $\mu(I^n)$ is eventually described by a polynomial—the Hilbert polynomial of the fibre ring of I . Eliahou, Herzog, and Saem [13] studied the question how small $\mu(I^2)$ can be in terms of $\mu(I)$ for a bivariate monomial ideal I . They construct examples where $\mu(I) > \mu(I^2)$, contrary to what one might have expected. For any given $n \in \mathbb{N}$, Abdolmaleki and Kumashiro [1] construct a bivariate monomial ideal I such that $\mu(I) > \mu(I^2) > \dots > \mu(I^n)$. Gasanova [19] shows that for every d there exists a monomial ideal I in any number of variables such that the inequality $\mu(I) > \mu(I^n)$ holds for any $n \leq d$.

While the unexpected behaviour of small powers is fascinating, our focus is set on large n . The asymptotic behaviour of the Hilbert function gives reason to suspect that eventually the actual set of minimal generators of I^n behaves well, in the sense that cancellations among the n -fold products of generators of I can be predicted.

For bivariate monomial ideals we describe the sets $G(I^n)$ of minimal generators of I^n explicitly for all n larger than a certain threshold. Specifically, we show that there exists s_0 such that for all $n \geq s \geq s_0$ every segment of the staircase diagram—and consequently, the set of minimal generators—of I^n is already determined by the staircase diagram of I^s . In other words, the staircase diagram of I^n can be build by aligning the staircase diagrams of certain subideals of I^s . We prove that

$$s_0 \leq \mu(I)(d^2 - 1) + 1,$$

where d is a constant depending on the degrees of the minimal generators of I which is at most the maximal x - or y -degree appearing in $G(I)$.

This chapter is structured as follows: Section 3.1 summarizes the necessary background about the integral closure of bivariate monomial ideals. In Section 3.2, we establish that the

¹This chapter is based on the submitted preprint [48] and is joint work with Roswitha Rissner.

corners of the Newton polyhedron of I play a special role among the minimal generators of powers of I . We call them persistent generators (Definition 3.2.1), as their powers remain minimal generators of all powers of I . The main result of this section is Theorem 3.2.14 which allows us to decompose powers of I into a sum of ideals:

$$I^{D+\ell} = \sum_{i=1}^k (g_i, g_{i+1})^\ell I^D,$$

where $g_1, \dots, g_{k+1}$ are the persistent generators of I , ordered in descending y -degree. We skip the details on how to calculate D here but note that $D \leq \mu(I) \cdot d$, where d is a constant depending on the degrees of the minimal generators of I and is at most the maximal x - or y -degree appearing in $G(I)$. We recover the fact [58, Proposition 2.1] that the ideal generated by the persistent generators is a reduction of I as a direct consequence of this theorem. In addition, it yields a bound for its reduction number, see Remark 3.2.12. Sums of that form for increasing ℓ are further studied in Section 3.3. We first describe the minimal generators of each summand $(g_i, g_{i+1})^\ell I^D$ separately (Theorem 3.3.4) and then add them up again (Theorem 3.3.16). Section 3.4 combines the results of Section 3.2 and Section 3.3 to the main results of this chapter (Theorem 3.4.6 and Corollary 3.4.11). We provide an implementation to compute $I^{s+\ell}$ in SageMath². We conclude the chapter with examples and runtime measurements in practice.

3.1 Integral closures of bivariate monomial ideals

For the basic definitions and facts about integral closures of monomial ideals, we refer to Section 2.2.2 in the preliminaries.

Throughout, we assume that I is not a principal ideal. Moreover, as developed in Section 2.1.5, when convenient for notation, we factor out the greatest common divisor of all monomials in I , and carry out computations with the shifted ideal $I_{\sharp} = I : \gcd(I)$, see Definition 2.1.46.

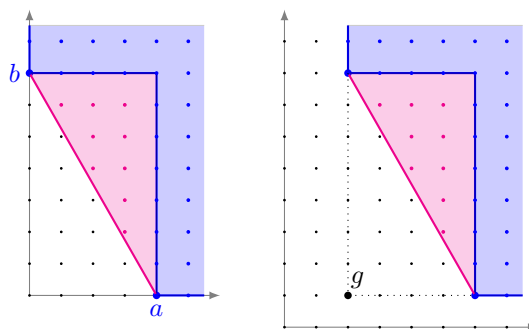


Fig. 3.1: **On the left**, we illustrate the integral closure of the ideal (x^a, y^b) . The integral closure includes all lattice points lying on or above the line segment connecting $(a, 0)$ and $(0, b)$, in addition to those already contained in the ideal. **On the right**, we present a visualisation of Fact 3.1.1(2).

²The program code associated with this chapter is available as ancillary file from the arXiv page of [48].

Fact 3.1.1 (cf. [32, Proposition 1.4.6]). For $a, b \in \mathbb{N}_0$ and a monomial $g \in k[x, y]$ the following are equivalent:

- (1) $x^u y^v \in \overline{(x^a, y^b)}$,
- (2) $g \cdot x^u y^v \in \overline{(g \cdot x^a, g \cdot y^b)}$,
- (3) $\frac{u}{a} + \frac{v}{b} \geq 1$.

Moreover, $\frac{u}{a} + \frac{v}{b} = 1$ implies $x^u y^v \in G(\overline{(x^a, y^b)})$.

Definition 3.1.2. Let $f, g, h \in k[x, y]$ be monomials such that $f \notin (g, h)$. We say that f lies between g and h if

$$\begin{aligned} \min\{\deg_x g, \deg_x h\} &< \deg_x f \text{ and} \\ \min\{\deg_y g, \deg_y h\} &< \deg_y f. \end{aligned}$$

Remark 3.1.3. Let f be a monomial that lies between two other monomials, g and h in $k[x, y]$. Geometrically, Fact 3.1.1 says that f is in the integral closure of (g, h) if and only if it lies above the line passing through g and h , see Figure 3.1. If f lies precisely on said line, then f is a minimal generator of the integral closure of (g, h) .

Definition 3.1.4. Let $I \subseteq k[x, y]$ be a monomial ideal. We define

$$\begin{aligned} \text{dist}_x I &:= \max\{\deg_x g \mid g \in G(I_\sharp)\} \text{ and} \\ \text{dist}_y I &:= \max\{\deg_y g \mid g \in G(I_\sharp)\}. \end{aligned}$$

We use $\text{dist}_\bullet$ as a placeholder for either dist_x or dist_y , with the choice remaining fixed within a given context. For a set of monomials $G \subseteq k[x, y]$, we define $\text{dist}_\bullet G$ as $\text{dist}_\bullet$ of the ideal generated by G .

Remark 3.1.5. We observe how dist behaves well additively and how $\text{dist}_\bullet I$ can be computed from the minimal generators of the ideal I .

- (1) Let f, g , and h be monomials such that f lies between g and h . Then $\text{dist}_\bullet(h, g)$ can be written as

$$\text{dist}_\bullet(h, g) = \text{dist}_\bullet(h, f) + \text{dist}_\bullet(f, g).$$

- (2) For a monomial ideal I , the following identity holds:

$$\text{dist}_\bullet I = \max\{\deg_\bullet g \mid g \in G(I)\} - \min\{\deg_\bullet g \mid g \in G(I)\}.$$

Note that if I is an anchored ideal, then the minimal x -degree and the minimal y -degree among the elements of $G(I)$ are both zero. Consequently $\min\{\deg_\bullet g \mid g \in G(I)\} = 0$.

Definition 3.1.6. Let g and h be two monomials. We define the non-standard grading $\deg_{g,h}$ additively by setting

$$\begin{aligned}\deg_{g,h}(x) &:= \text{dist}_y(g, h) \text{ and} \\ \deg_{g,h}(y) &:= \text{dist}_x(g, h).\end{aligned}$$

Further, we set $d_{g,h} := \deg_{g,h}(g) = \deg_{g,h}(h)$.

Remark 3.1.7. (1) For $a, b \in \mathbb{N}$, the identity $d_{x^a, y^b} = ab$ holds.

(2) Let g and h be two monomials, and let $x^\alpha y^\beta := \gcd(g, h)$. Then

$$\deg_{g,h}(x^u y^v) \geq d_{g,h} \iff \frac{u - \alpha}{\text{dist}_x(g, h)} + \frac{v - \beta}{\text{dist}_y(g, h)} \geq 1,$$

where equality holds on both sides simultaneously. Note that here $x^u y^v$ need not lie between g and h .

(3) Geometrically, the equivalence in (2) states $\deg_{g,h}(x^u y^v) \geq d_{g,h}$ if and only if $x^u y^v$ lies on or above the line passing through g and h . Equality holds precisely when $x^u y^v$ lies on the line.

Lemma 3.1.8. Let $f, g, h \in k[x, y]$ be monomials such that f lies between g and h . Then the following assertions are equivalent:

- (1) $f \in \overline{(g, h)}$,
- (2) $\deg_{g,h}(f) \geq d_{g,h}$,
- (3) $\deg_{f,h}(g) \leq d_{f,h}$,
- (4) $\deg_{g,f}(h) \leq d_{g,f}$.

Moreover, $\deg_{g,h}(f) = d_{g,h}$ implies $f \in G(\overline{(g, h)})$.

Proof. (1) $\Leftrightarrow$ (2) follows from Remark 3.1.7(1) and Fact 3.1.1. (2) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4) follow from Remark 3.1.7(3). $\square$

3.2 The role of persistent generators

Definition 3.2.1. Let I be a monomial ideal in $k[x, y]$. We say $f \in G(I)$ is a **persistent generator of I** if $f \notin \overline{(g, h)}$ for all monomials $g, h \in I \setminus \{f\}$. We denote the set of all persistent generators of I by $P(I)$, and define $N(I) := G(I) \setminus P(I)$.

Remark 3.2.2. (1) The minimal generators of I with maximal x -degree and y -degree, respectively, are persistent.

(2) The persistent generators of I are the corners of the Newton polyhedron of I .

(3) If $f \in P(I)$, then $f^n \in G(I^n)$ for any $n \in \mathbb{N}$.

By Fact 2.2.14, a monomial f is an element of $\overline{(g, h)}$ if and only if $f^n \in (g, h)^n$ for some n . In the next proposition, we explicitly determine such an n . In addition, we show that if $f \notin \overline{(g, h)}$, an analogous relation holds among the three polynomials.

Proposition 3.2.3. *Let $g, h, f \in k[x, y]$ with f lying between g and h , and define*

$$\alpha := \text{dist}_\bullet(f, h) \quad \text{and} \quad n := \text{dist}_\bullet(g, h).$$

Then the following assertions hold:

(1) *If $f \in \overline{(g, h)}$ then $g^\alpha h^{n-\alpha} \mid f^n$ and hence $f^n \in (g, h)^n$.*

(2) *If $f \notin \overline{(g, h)}$ then $f^n \mid g^\alpha h^{n-\alpha}$ and hence $g^\alpha h^{n-\alpha} \in (f)^n$.*

Moreover, $\deg_{g,h}(f) = d_{g,h}$ if and only if there exist $n \in \mathbb{N}$ and $0 \leq k \leq n$ such that $g^k h^{n-k} = f^n$.

Proof. Recall from Remark 3.1.5(1) that $n - \alpha = \text{dist}_\bullet(g, f)$. Since f lies between g and h it follows that all three monomials are divisible by $\gcd(g, h)$. Therefore, we can assume that $g = y^b$, $h = x^a$ and $f = x^u y^v$ with $0 < u < a$ and $0 < v < b$. Note that

$$\deg_x(x^{au} y^{b(a-u)}) = au = \deg_x(f^a).$$

By Fact 3.1.1, $f \in \overline{I}$ if and only if $\frac{u}{a} + \frac{v}{b} \geq 1$ which, in turn, is equivalent to

$$\deg_y(x^{au} y^{b(a-u)}) = b(a-u) \leq av = \deg_y(f^a).$$

This implies (1) and (2) in the case that $\text{dist}_\bullet = \text{dist}_x$. The respective assertions with $\text{dist}_\bullet = \text{dist}_y$ are proven analogously.

For the last assertion note that the right-hand side holds if and only if

$$ak = un \quad \text{and} \quad b(n-k) = vn,$$

which is equivalent to $\frac{u}{a} + \frac{v}{b} = 1$. The latter is equivalent to the left-hand side, cf. Remark 3.1.7(2). $\square$

Remark 3.2.4. Note that in the second assertion of Proposition 3.2.3, the assumption that $f \notin \overline{(g, h)}$ implies that the equality $\deg_{g,h}(f) = d_{g,h}$ cannot hold. In particular, we have that $f^n \neq g^\alpha h^{n-\alpha}$.

Remark 3.2.5. The bound $\min\{\text{dist}_x(g, h), \text{dist}_y(g, h)\}$ for n in Proposition 3.2.3 is sharp, see Example 3.2.6 below.

Example 3.2.6. Let $g = y^5$ and $h = x^6$. Then

$$\min\{\text{dist}_x(g, h), \text{dist}_y(g, h)\} = 5.$$

- (1) The monomial $f = x^5y$ is an element of $\overline{(g, h)}$ and one can easily verify that $f^n \notin (x^6, y^5)^n$ holds for $n \leq 4$.
- (2) The monomial $f = xy^4$ lies between g and h and $f \notin \overline{(g, h)}$. A straight-forward computation shows that for all $n \in \{1, 2, 3, 4\}$ there exist no $\alpha, \beta \in \mathbb{N}$ such that $\alpha + \beta = n$, and $g^\alpha h^\beta \in (f)^n$.

Definition 3.2.7. Let I be a monomial ideal in $\mathbb{k}[x, y]$. We say that $f \in G(I)$ is **weakly persistent**, if $f^n \in G(I^n)$ holds for all $n \in \mathbb{N}$. We denote by $P^*(I)$ the set of all weakly persistent generators of I .

Remark 3.2.8. Clearly, the inclusion $P(I) \subseteq P^*(I)$ holds. Let $g_1, \dots, g_{k+1}$ be the persistent generators of I , ordered in descending y -degree. Then Proposition 3.2.3 implies that

$$P^*(I) = P(I) \uplus \left\{ f \in N(I) \mid \deg_{g_i, g_{i+1}}(f) = d_{g_i, g_{i+1}} \text{ for some } i \in [k] \right\}.$$

Notation 3.2.9. Let $F = \{g_1, \dots, g_{k+1}\}$ be a set of monomials such that $g_1, \dots, g_{k+1}$ are ordered in descending y -degree. We set

$$\delta_F := \max_{1 \leq i \leq k} \left\{ \min\{\text{dist}_x(g_i, g_{i+1}), \text{dist}_y(g_i, g_{i+1})\} - 1 \right\}.$$

Corollary 3.2.10. Let I be a monomial ideal in $\mathbb{k}[x, y]$ and let $P(I) \subseteq P \subseteq P^*(I)$ such that $P = \{g_1, \dots, g_{k+1}\}$ and $g_1, \dots, g_{k+1}$ are ordered in descending y -degree. For every $n \geq \delta_P$ and $f \in G(I) \setminus P$ there exist $1 \leq i \leq k$ and $a \leq \delta_P$ such that

$$f^n \in (g_i, g_{i+1})^{n-a} \cdot f^a.$$

Proof. Note that $G(I) \setminus P \subseteq N(I)$. By definition, there exist $g, h \in I \setminus \{f\}$ with $f \in \overline{(g, h)}$. We can choose g and h to be in P such that no other element of P lies between them, meaning $g = g_i$ and $h = g_{i+1}$ for some $1 \leq i \leq k$. We write $d := \min\{\text{dist}_x(g_i, g_{i+1}), \text{dist}_y(g_i, g_{i+1})\}$. Then $n \geq \delta_P \geq d - 1$, so we can write $n = qd + a$ with $q \in \mathbb{N}_0$ and $a \leq d - 1$. It follows from Proposition 3.2.3(1) that

$$f^n = f^{qd+a} \in (g_i, g_{i+1})^{qd} \cdot f^a.$$

□

Corollary 3.2.10 immediately yields:

Corollary 3.2.11. *Let $I \subseteq k[x, y]$ be a monomial ideal, $n \in \mathbb{N}$, and let $P(I) \subseteq P \subseteq P^*(I)$. Then every minimal generator of I^n is of the form*

$$\prod_{g \in P} g^{\ell_g} \cdot \prod_{f \in G(I) \setminus P} f^{k_f},$$

where $0 \leq k_f \leq \delta_P$ for all $f \in G(I) \setminus P$ and $\sum_g \ell_g + \sum_f k_f = n$.

Remark 3.2.12. Singla [58, Proposition 2.1] established that the ideal $\mathfrak{a}$, generated by the persistent generators $P(I)$, is a reduction of I . That is, there exists $\delta \in \mathbb{N}$ such that for all $n \in \mathbb{N}_0$,

$$I^{\delta+n} = \mathfrak{a}^n I^\delta.$$

Singla's result applies to monomial ideals in any number of variables. Through Corollary 3.2.11 we recover Singla's result for the bivariate case and further show that the reduction number of I with respect to $\mathfrak{a}$ is at most $|G(I) \setminus P(I)| \cdot \delta_{P(I)}$.

With Proposition 3.2.3(2), we further refine the statement of Corollary 3.2.11 in Theorem 3.2.14 below.

Notation 3.2.13. For a monomial ideal I and $P(I) \subseteq P \subseteq P^*(I)$. We set

$$d_P := \begin{cases} \min\{\text{dist}_x I, \text{dist}_y I\} - 2, & \text{if } |P| > 2, \\ 0, & \text{if } |P| = 2. \end{cases}$$

Note that $|P| > 2$ implies that $\min\{\text{dist}_x I, \text{dist}_y I\} \geq 2$.

Theorem 3.2.14. *Let I be a monomial ideal in $k[x, y]$ and let $P(I) \subseteq P \subseteq P^*(I)$ such that $P = \{g_1, \dots, g_{k+1}\}$ and $g_1, \dots, g_{k+1}$ are ordered in descending y -degree. Further, let $D \geq (\mu(I) - |P|) \cdot \delta_P + |P| \cdot d_P$.*

Then for all $\ell \geq 0$,

$$I^{D+\ell} = \sum_{i=1}^k (g_i, g_{i+1})^\ell I^D.$$

Proof. The inclusion " $\supseteq$ " is trivial. We prove " $\subseteq$ ". Write $N := G(I) \setminus P$. By Corollary 3.2.11 we can write $I^{D+\ell} = I^{\delta+n} = \mathfrak{a}^n I^\delta$ where $\delta := |N| \cdot \delta_P$, and $n := (k+1)d_P + \ell$, and $\mathfrak{a}$ is the ideal generated by P . Thus, every minimal generator F of $I^{\delta+n}$ is of the form $F = g \cdot f$, where g is a product of n elements in P and $f \in I^\delta$.

Claim. There exists $1 \leq i \leq k$ such that

$$g \in (g_i, g_{i+1})^\ell \mathfrak{a}^{(k+1)d_P}.$$

We write $g = g_1^{n_1} \cdots g_{k+1}^{n_k}$, where $n_i \in \mathbb{N}_0$ with $\sum_i n_i = n$. The assertion of the claim holds trivially in the following cases:

(1) at most one $n_i > d_P$,

(2) $n_j, n_{j+1} > d_P$ for some $1 \leq j \leq k$ and $n_i \leq d_P$ for all $i \notin \{j, j+1\}$.

Otherwise, we take $a := \min\{i \mid n_i > d_P\}$ and $b := \max\{i \mid n_i > d_P\}$. Note that $a + 1 < b$. The following argument may be repeated as needed; in every step either a increases strictly or b decreases strictly. For readability, we may therefore assume without loss of generality that $a = 1$ and $b = k + 1$ at the outset. By Proposition 3.2.3(2), we then have

$$g_2^{\text{dist}_\bullet(g_1, g_{k+1})} \mid g_1^{\text{dist}_\bullet(g_2, g_{k+1})} g_{k+1}^{\text{dist}_\bullet(g_1, g_2)}.$$

Since g is a minimal generator, equality must hold, so

$$g = g_1^{n_1 - \text{dist}_\bullet(g_2, g_{k+1})} g_2^{n_2 + \text{dist}_\bullet(g_1, g_{k+1})} g_3^{n_3} \cdots g_k^{n_k} g_{k+1}^{n_{k+1} - \text{dist}_\bullet(g_1, g_2)}.$$

We iteratively apply Proposition 3.2.3(2) until at least one of the exponents of g_1 and g_2 is less than or equal to d_P . At this stage, we redefine $a' := \min\{i \mid n_i > d_P\}$ and $b' := \max\{i \mid n_i > d_P\}$. Now $a' > a$ or $b' < b$ must hold. Hence, by repeating this argument from the top we eventually must reach one of the trivial cases (1) or (2), where the claim follows immediately. $\square$

Remark 3.2.15. The set P may be chosen closer to either $P(I)$ or $P^*(I)$, depending on the specific context in which Theorem 3.2.14 is applied. For instance, if the objective is to minimize $(\mu(I) - |P|) \cdot \delta_P + |P| \cdot d_P$, then P can be selected based on the values of d_P and δ_P .

3.3 Ideals with regular staircase factors

In this section, we study the minimal generators of sums of ideals of the form $(g, h)^n J$, where g and h are monomials, and J is a (fixed) anchored monomial ideal in $k[x, y]$. The main result of this section is Theorem 3.3.16.

We begin with the special case $(x^u, y^v)^n J$ for $u, v \in \mathbb{N}$. By drawing its exponents in the xy -plane, the ideal $(x^u, y^v)^n J$ looks like a “regular staircase” in the sense that all n steps in its staircase are of the same size. The minimal generators

of the product $(x^u, y^v)^n J$ are of the form $x^{u(n-i)} y^{vi} f$ with $f \in G(J)$. In general, not all elements of this form are minimal generators as divisibility relations may occur among them. In Figure 3.2 we visualize the potential cancellations among the minimal generators in the product $(x^u, y^v)^n J$. Even though Figure 3.2 does not depict the actual generators of the ideal J , the figure suggests that the “overlaps” of the shifted copies of J result in a repeating pattern with increasing n . We formalize this “pattern repetition” in Theorem 3.3.4 below. Before that, we establish in Lemma 3.3.2 that partitioning the elements of $(x^u, y^v)^n J$ based on their y -degrees reveals divisibilities by certain powers of x^u or y^v . Figure 3.3

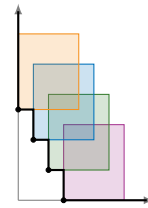


Fig. 3.2

visualizes this partition.

Remark 3.3.1. The choice of partitioning by y -degree is arbitrary; all results of this section remain valid if we instead partition by x -degree, simply by interchanging roles of x and y .

Lemma 3.3.2. Let $u, v \in \mathbb{N}$ and $J \subseteq k[x, y]$ be an anchored monomial ideal. Moreover, let $r \geq \left\lceil \frac{\text{dist}_y J}{v} \right\rceil$ and $n \geq r$. For $r \leq j \leq n$, we set

$$\mathcal{U}_j = \{F \in (x^u, y^v)^n J \mid \deg_y F \geq jv\} \text{ and}$$

$$\mathcal{L}_j = \{F \in (x^u, y^v)^n J \mid \deg_y F < jv\}.$$

Then, for $r \leq j \leq n$,

$$\mathcal{U}_j \subseteq y^{v(j-r)} \cdot (x^u, y^v)^{n-(j-r)} J, \quad \text{and} \quad \mathcal{L}_j \subseteq x^{u(n-j)} \cdot (x^u, y^v)^j J.$$

In particular, for $r \leq j \leq n-1$,

$$\mathcal{U}_j \cap \mathcal{L}_{j+1} = x^{u(n-(j+1))} y^{v(j-r)} \cdot \{f \in (x^u, y^v)^{r+1} J \mid rv \leq \deg_y f < (r+1)v\}.$$

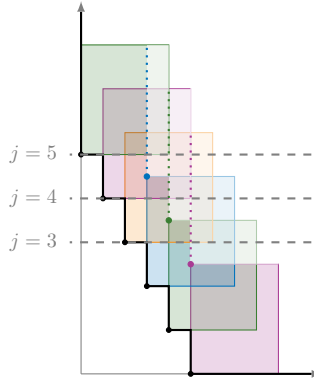


Fig. 3.3: We partition the ideal $(x^u, y^v)^n J$ into sections based on the y -degree, as indicated by the dashed horizontal lines. Observe that the upper left corner of each rectangle corresponds to a monomial in $(x^u, y^v)^n J$ (we assumed J to be anchored). This bounds the x -degree in each y -section; see Remark 3.3.3.

Proof. For the first two inclusions, we write $F = x^{u(n-i)} y^{vi} \tilde{f}$ with $0 \leq i \leq n$ and $\tilde{f} \in J$ and separate into two cases.

$\mathcal{U}_j$: Note that $\deg_y \tilde{f} \leq \text{dist}_y J$. Hence, $\deg_y F \geq jv$ implies $i \geq j - \frac{\text{dist}_y J}{v} \geq j - r \geq 0$, that is, $F = y^{v(j-r)} \cdot f$ with $f = x^{u(n-i)} y^{v(i-(j-r))} \tilde{f} \in (x^u, y^v)^{n-(j-r)} J$.

$\mathcal{L}_j$: The condition $\deg_y F < jv$ implies $i \leq j$ and $n-i \geq n-j \geq 0$. Therefore, $F = x^{u(n-j)} \cdot f$ with $f = x^{u(j-i)} y^{vi} \tilde{f} \in (x^u, y^v)^j J$.

Finally, for the last equality, “ $\subseteq$ ” follows from the above while “ $\supseteq$ ” is obvious. $\square$

Remark 3.3.3. As preparation for later arguments, we provide a bound on the x -degrees of elements in $\mathcal{U}_j$: Let J be an anchored monomial ideal with $b := \text{dist}_y J$. With the notation of Lemma 3.3.2, if $n > j - r$ and $j \geq r$, then $H := x^{u(n-j+r)}y^{v(j-r)}y^b$ is an element of $(x^u, y^v)^n J$ satisfying

$$\deg_y H = v(j - r) + b \leq jv \quad \text{and} \quad \deg_x H = (n - j + r)u,$$

cf. Figure 3.3. Consequently, for $n > j - r$,

$$f \in \mathcal{U}_j \cap G((x^u, y^v)^n J) \implies \deg_x f \leq (n + r - j)u,$$

and equality can only hold if $H = f$. In particular, if there exists $f \in \mathcal{U}_j \cap G((x^u, y^v)^n J)$ with $\deg_x f = (n + r - j)u$, then $H = f \in \mathcal{U}_j$ which, considering the y -degree of H , further implies $r = \frac{b}{v}$ and $\deg_y(f) = \text{dist}_y(J)$.

Theorem 3.3.4. Let $u, v \in \mathbb{N}$ and $J \subseteq k[x, y]$ be an anchored monomial ideal.

Then, for all $r \geq \left\lceil \frac{\text{dist}_y J}{v} \right\rceil$ and $\ell \in \mathbb{N}_0$,

$$G((x^u, y^v)^{r+1+\ell} J) = y^{v\ell} L \uplus \biguplus_{j=1}^{\ell} x^{uj} y^{v(\ell-j)} M \uplus x^{u\ell} R,$$

where

$$\begin{aligned} L &= \{f \in G((x^u, y^v)^{r+1} J) \mid \deg_y f \geq rv\}, \\ M &= \{f \in G((x^u, y^v)^{r+1} J) \mid rv \leq \deg_y f < (r+1)v\}, \text{ and} \\ R &= \{f \in G((x^u, y^v)^{r+1} J) \mid \deg_y f < rv\}. \end{aligned}$$

In particular,

$$\mu((x^u, y^v)^{r+1+\ell} J) = \mu((x^u, y^v)^{r+1} J) + \ell \cdot |M|.$$

Proof. Note that the count is an immediate consequence of the first assertion.

With the notation of Lemma 3.3.2, we have

$$(x^u, y^v)^{r+1+\ell} J = \mathcal{L}_r \uplus \biguplus_{j=r}^{r+\ell-1} (\mathcal{U}_j \cap \mathcal{L}_{j+1}) \uplus \mathcal{U}_{r+\ell}. \quad (3.3.1)$$

We claim that the following three statements hold:

- (1) $\mathcal{U}_{r+\ell} \cap G((x^u, y^v)^{r+1+\ell} J) = y^{v\ell} \cdot L$,
- (2) $\mathcal{L}_r \cap G((x^u, y^v)^{r+1+\ell} J) = x^{u\ell} \cdot R$, and

(3) for all $r \leq j \leq r + \ell - 1$,

$$(\mathcal{U}_j \cap \mathcal{L}_{j+1}) \cap G((x^u, y^v)^{r+1+\ell} J) = x^{u(r+\ell-j)} y^{v(j-r)} \cdot M.$$

If the claim holds, then the assertion follows from (3.3.1) since, arranging the sets in reverse order,

$$G((x^u, y^v)^{r+1+\ell} J) \cap \biguplus_{j=r}^{r+\ell-1} (\mathcal{U}_j \cap \mathcal{L}_{j+1}) = \biguplus_{j=1}^{\ell} x^{uj} y^{v(\ell-j)} \cdot M.$$

In all three cases, the inequalities that the y -degrees must satisfy, are the same on both sides of the equality. Moreover, the inclusions " $\subseteq$ " all hold due to Lemma 3.3.2 and the fact that a generator $g \cdot f$ of $(x^u, y^v)^{r+1+\ell} J$ with $g \in (x^u, y^v)^{r+1+\ell}$ and $f \in J$ can only be minimal, provided that f is a minimal generator of J (the same holds for g but is not relevant here).

For the reverse inclusions, it is in all three cases left to show that every element of the set on the right is a minimal generator of $(x^u, y^v)^{r+1+\ell} J$. To do so, take

$$H = x^{u(r+1+\ell-i)} y^{vi} h \in (x^u, y^v)^{r+1+\ell} J$$

with $h \in J$ and $0 \leq i \leq \ell + r + 1$.

(1) Let $F = y^{v\ell} f$ with $f \in G((x^u, y^v)^{r+1} J)$ and $\deg_y f \geq rv$. Then $\deg_x F = \deg_x f \leq (r+1)u$, cf. Remark 3.3.3. If H divides F , then $\deg_x H \leq \deg_x F$ implies $(r+1+\ell-i)u \leq (r+1)u$, that is, $i \geq \ell$. We cancel out $y^{v\ell}$ to conclude that

$$x^{u(r+1+\ell-i)} y^{v(i-\ell)} h \mid f,$$

which, since $f \in G((x^u, y^v)^{r+1} J)$, implies that $H = F$.

(2) Let $F = x^{u\ell} f$ with $f \in G((x^u, y^v)^{r+1} J)$ and $\deg_y f < rv$. Again, assume that H divides F . Then $\deg_y H = iv + \deg_y h < rv$ and hence $i < r$. This implies that $r + \ell + 1 - i > \ell + 1$, thus we can cancel out $x^{u\ell}$ on both sides and end up with the same conclusion as in (1).

(3) Let $r \leq j \leq r + \ell - 1$ and $F = x^{u(r+\ell-j)} y^{v(j-r)} f$ with $f \in M$. Then $f \in L$, thus by (1), $y^{v(j-r)} f$ is a minimal generator of $(x^u, y^v)^{j+1} J$. As above, if $H \mid F$, then the y -degree of H must be less or equal than the y -degree of F , and therefore $i \leq j + 1$. This implies $r + \ell + 1 - i \geq r + \ell - j$, so we can cancel out $x^{u(r+\ell-j)}$ which leaves us with

$$x^{u(j-i+1)} y^{vi} h \mid y^{v(j-r)} f.$$

As $y^{v(j-r)} f$ is minimal in $(x^u, y^v)^{j+1} J$, we get equality. □

Let us unravel Theorem 3.3.4 in an example.

Example 3.3.5. Let $u = 3$, $v = 4$, and

$$J = (y^{10}, x^2y^7, x^3y^5, x^5y^4, x^7y^2, x^9)$$

as depicted in the (grey) rectangle in Figure 3.4. We choose $r = 3$, which is the minimal possible choice of r in Theorem 3.3.4. The lower two dashed lines in the figure mark the areas where the y -degree is in between rv and $(r+1)v$.

The left part of Figure 3.4 shows $(x^3, y^4)^{r+1}J$. The set L consists of generators above the line $j = r$, marked with (blue) circles. The set R consists of the (orange) squares below $j = r$. The middle set M contains the two encircled (in red) generators between the two lines $j = r$ and $j = r + 1$.

On the right side of Figure 3.4, we see $(x^3, y^4)^{r+3}J$. The minimal generators are a disjoint union of the sets

$$y^{4 \cdot 2}L, \quad x^3y^4M, \quad x^{3 \cdot 2}M, \quad \text{and} \quad x^{3 \cdot 2}R.$$

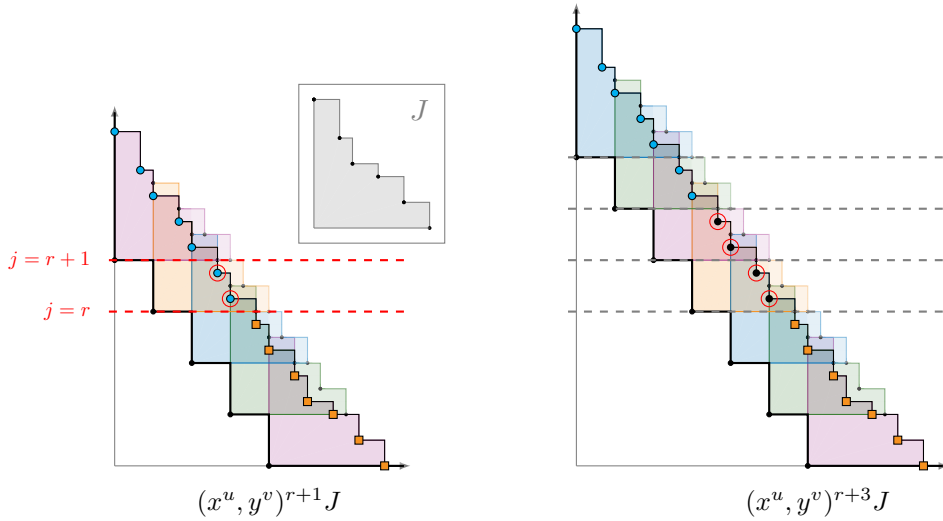


Fig. 3.4: Visualisation of the sets L , M , and R in Example 3.3.5.

Corollary 3.3.6. Let $u, v \in \mathbb{N}$, J be an anchored monomial ideal, $r \geq \left\lceil \frac{\text{dist}_y J}{v} \right\rceil$, and

$$M := \left\{ f \in G\left((x^u, y^v)^{r+1}J\right) \mid rv \leq \deg_y f < (r+1)v \right\}.$$

Then $M \neq \emptyset$.

Proof. Let $b := \text{dist}_y J$ and choose any natural number $\ell > \frac{b}{v}$. Assume that $M = \emptyset$. Then, by (3) in the proof of Theorem 3.3.4, we have

$$(\mathcal{U}_j \cap \mathcal{L}_{j+1}) \cap G\left((x^u, y^v)^{r+\ell+1}J\right) = \emptyset$$

for all $r \leq j \leq r + \ell + 1$. Consequently, there are no minimal generators of $(x^u, y^v)^{r+\ell+1}J$ satisfying

$$rv \leq \deg_y f < (r + \ell)v. \quad (3.3.2)$$

However, by the choice of ℓ , the element $f := y^{rv}x^{(\ell+1)u}y^b \in (x^u, y^v)^{r+\ell+1}J$ fulfills these inequalities. Therefore, there must exist a minimal generator $g \in G((x^u, y^v)^{r+\ell+1}J)$ dividing f . Then $\deg_x g \leq (\ell + 1)u$ and by (3.3.2), $\deg_y g < rv$ must hold, and hence $g \notin (x^u, y^v)^{r+\ell+1}J$, a contradiction. $\square$

Theorem 3.3.4 describes how the minimal generators of $(x^u, y^v)^{r+1+\ell}J$ change in a predictable pattern as ℓ increases. In other words, the staircase of the ideal $(x^u, y^v)^{r+1+\ell}J$ is formed by aligning the staircases of certain repeatedly occurring subideals: from left to right, we begin with the staircase of (L) , then we repeat the staircase of (M) for ℓ consecutive steps, and finally add the staircase of (R) .

We want to formalize this idea of “connecting” staircases.

Definition 3.3.7. For $I, J \subseteq k[x, y]$ monomial ideals, we define the **link** $I \circledast J$ (with respect to y) as

$$I \circledast J := I_{\mathfrak{J}} \cdot y^{\text{dist}_y J} + J_{\mathfrak{J}} \cdot x^{\text{dist}_x I},$$

and we write $I^{\circledast \ell} := \underbrace{I \circledast I \circledast \cdots \circledast I}_{\ell}$. We call the monomial $x^{\text{dist}_x I} y^{\text{dist}_y J}$ the **link point** of $I \circledast J$.

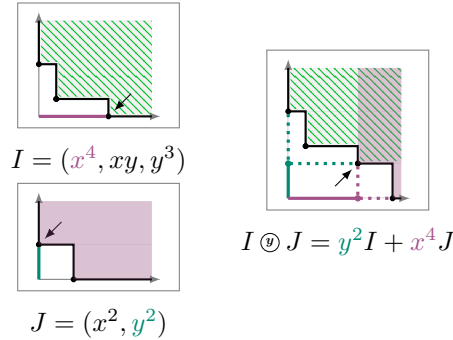


Fig. 3.5: Visualisation of the link of ideals.

Remark 3.3.8. As mentioned above in Remark 3.3.1, we want to be able to reverse the roles of x and y throughout. However, this affects the order of the arguments of the link. Note that $I \circledast J = J \circledast I$, which is why we have to include the variable used for the partition in the notation of the link.

Remark 3.3.9. If I and J are monomial ideals with $\text{dist}_x I = a$ and $\text{dist}_y J = b$, then the minimal generators of $I_{\mathfrak{J}} \cdot y^b$ and $J_{\mathfrak{J}} \cdot x^a$ only intersect in one element, namely the link

point $x^a y^b$, cf. Figure 3.5 (indicated by an arrow). Therefore,

$$\begin{aligned} G(I \circledast J) &= G(I_{\sharp}) \cdot y^b \cup G(J_{\sharp}) \cdot x^a \\ &= G(I_{\sharp}) \cdot y^b \uplus \left(G(J_{\sharp}) \setminus \{y^b\} \right) \cdot x^a. \end{aligned}$$

This implies $\mu(I \circledast J) = \mu(I) + \mu(J) - 1$.

Remark 3.3.10. Let I be an anchored monomial ideal that can be written as the link of ideals, i.e.,

$$I = J_0 \circledast \cdots \circledast J_k$$

for some anchored monomial ideals $J_0, \dots, J_k$. For $1 \leq i \leq k$ we denote the link point between $(J_0 \circledast \cdots \circledast J_{i-1})$ and $(J_i \circledast \cdots \circledast J_k)$ with h_i , and we set $h_0 := y^{\text{dist}_y I}$ and $h_{k+1} := x^{\text{dist}_x I}$. Then it follows that

$$I = \sum_{j=0}^k \gcd(h_i, h_{i+1}) J_i$$

and, in particular, $I : \gcd(h_i, h_{i+1}) = J_i$ for all $0 \leq i \leq k$.

With the necessary tools in place, we now return to the goal of expressing $(x^u, y^v)^{r+1+\ell} J$ as the link of the ideals generated by L , M , and R . However, there are “gaps” between the staircases of these ideals, so we must first expand them by suitable link points.

Definition 3.3.11. Let $u, v \in \mathbb{N}$, $J \subseteq k[x, y]$ be an anchored monomial ideal, and $r \geq \left\lceil \frac{\text{dist}_y J}{v} \right\rceil$. Further, we set $g := x^\alpha y^\beta$ to be the minimal generator of $(x^u, y^v)^{r+1} J$ with

$$\beta = \min \left\{ \deg_y f \mid f \in G\left((x^u, y^v)^{r+1} J\right), \deg_y f \geq rv \right\}.$$

With L , M , and R as in Theorem 3.3.4 we define the r -**segments** of $(x^u, y^v)J$ as

$$\begin{aligned} A &:= (L) : y^\beta, \\ H &:= \left(M \cup \{x^{\alpha-u} y^{\beta+v}\} \right) : x^{\alpha-u} y^\beta, \text{ and} \\ B &:= (R \cup \{g\}) : x^\alpha, \end{aligned}$$

For monomials $g, h \in k[x, y]$ we define the r -segments of $(g, h)J$ as the r -segments of $(g, h)_{\sharp} J$.

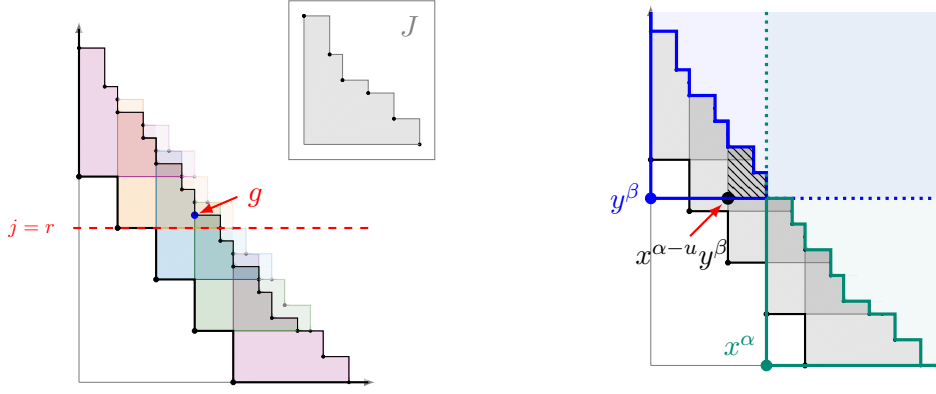


Fig. 3.6: **Left:** Visualisation of the point g from Definition 3.3.11, with $r = 3$. **Right:** Above y^β is the staircase of $y^\beta A$ (in blue), and to the right of x^α is the staircase of $x^\alpha B$ (in green). The striped area in the bottom-right corner of $y^\beta B$ is the staircase of $x^{\alpha-u} y^\beta H$.

Remark 3.3.12. With the notation of Definition 3.3.11, we have

$$\begin{aligned} A &= ((x^u, y^v)^{r+1} J) : y^\beta, \\ H &= ((x^u, y^v)^{r+1} J) : x^{\alpha-u} y^\beta, \text{ and} \\ B &= ((x^u, y^v)^{r+1} J) : x^\alpha. \end{aligned}$$

In particular, $g = x^\alpha y^\beta$ is the link point of $A \oplus B = (x^u, y^v)^{r+1} J$, see Figure 3.6. Moreover,

$$L = G(A)y^\beta, \quad R = G(B)x^\alpha \setminus \{g\}, \quad \text{and} \quad M = G(H)x^{\alpha-u} y^\beta \setminus \{x^{\alpha-u} y^{\beta+v}\}.$$

Corollary 3.3.13. Let $u, v \in \mathbb{N}$ and $J \subseteq k[x, y]$ be an anchored monomial ideal, $r \geq \left\lceil \frac{\text{dist}_y J}{v} \right\rceil$, and A, H, B the r -segments of $(x^u, y^v)J$.

Then, for all $\ell \in \mathbb{N}_0$,

$$(x^u, y^v)^{r+1+\ell} J = A \oplus H^{\oplus \ell} \oplus B.$$

Proof. It follows from Theorem 3.3.4 in combination with the last equalities in Remark 3.3.12 that

$$\begin{aligned} G((x^u, y^v)^{r+1+\ell} J) &= y^{v\ell} L \oplus \bigoplus_{j=1}^{\ell} x^{ju} y^{(\ell-j)v} M \oplus x^{u\ell} R \\ &= y^{v\ell+\beta} G(A) \oplus \bigoplus_{j=0}^{\ell-1} (x^{\alpha+ju} y^{\beta+(\ell-j-1)v} G(H) \setminus \{x^{\alpha+ju} y^{\beta+(\ell-j)v}\}) \\ &\quad \oplus x^{\alpha+u\ell} G(B) \setminus \{x^{\alpha+u\ell} y^\beta\}. \end{aligned}$$

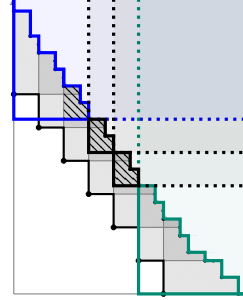
This is exactly the minimal generating set of

$$A \oplus H^{\oplus \ell} \oplus B = Ay^{v\ell+\beta} + \sum_{j=0}^{\ell-1} Hx^{\alpha+ju} y^{\beta+(\ell-1-j)v} + Bx^{\alpha+u\ell},$$

cf. Remark 3.3.9. Note that the generators $x^{\alpha+j u} y^{\beta+(\ell-j) v}$ for $0 \leq j \leq \ell$ are the link points in $A \circledast H^{\circledast \ell} \circledast B$. $\square$

Remark 3.3.14. Corollary 3.3.13 confirms that r -segments behave as intended and rephrases Theorem 3.3.4 in the language of ideal links. Continuing Figures 3.4 and 3.6, we provide a visualisation in Figure 3.7.

Fig. 3.7: In $(x^u, y^v)^{r+3} J$, the staircase of H (striped black) is repeated $\ell = 2$ times in the middle. On the top left (in blue) is the staircase of A and on the bottom right (in green) is the staircase of B . Note that there is one H included in A .



Remark 3.3.15. With the notation of Definition 3.3.11,

$$\beta < (r+1)v \quad \text{and} \quad \alpha \leq (r+1)u$$

holds. The first inequality follows from Corollary 3.3.6, while the second follows from Remark 3.3.3, since by definition, $g = x^\alpha y^\beta$ is a minimal generator of $(x^u, y^v)^{r+1} J$ that is in $\mathcal{U}_r$. As noted in Remark 3.3.3, if we have equality $\alpha = (r+1)u$, then this implies $rv = \text{dist}_y J$, $g = x^{(r+1)u} y^{\text{dist}_y(J)}$, and $\beta = \text{dist}_y(J)$.

We are now set to prove the main result of this section.

Theorem 3.3.16. Let $I \subseteq k[x, y]$ be an anchored monomial ideal such that $G(I) = P^*(I) = \{g_1, \dots, g_{k+1}\}$ and the g_i are ordered in descending y -degree. Further, let $J \subseteq k[x, y]$ be an anchored monomial ideal, and for $1 \leq i \leq k$, let $v_i := \text{dist}_y(g_i, g_{i+1})$, $u_i := \text{dist}_x(g_i, g_{i+1})$, and

$$r \geq \left\lceil \max_{1 \leq i \leq k} \left\{ \frac{\text{dist}_y J}{v_i} \right\} \right\rceil.$$

Then, for all $\ell \geq 0$,

$$\sum_{i=1}^k (g_i, g_{i+1})^{r+1+\ell} J = C_0 \circledast \left(\bigcirc_{i=1}^k H_i^{\circledast \ell} \circledast C_i \right),$$

where, for $1 \leq i < k$, A_i, H_i, B_i are the r -segments of $(g_i, g_{i+1})J$,

$$C_i := B_i \cdot y^{(r+1)v_{i+1} - \text{dist}_y(B_{i+1})} + A_{i+1} \cdot x^{(r+1)u_i - \text{dist}_x(A_i)},$$

$C_0 := A_1$, and $C_k := B_k$.

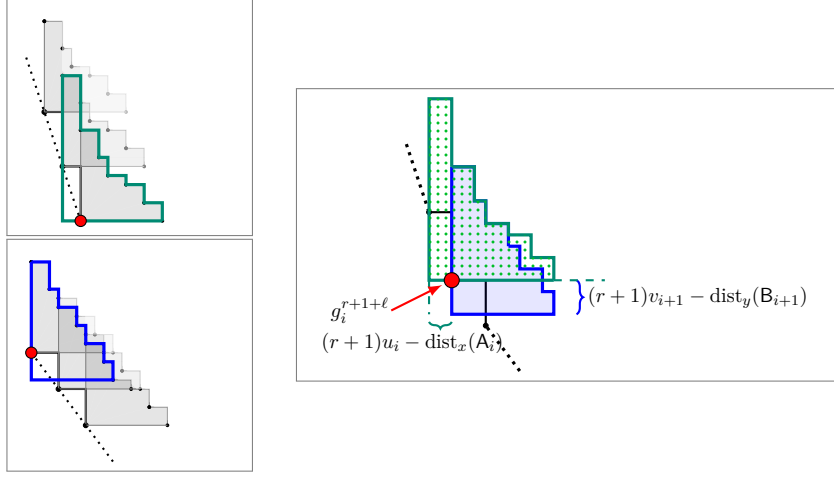


Fig. 3.8: **Left:** The lower section of $(g_{i-1}, g_i)^{r+1+\ell} J$ and the upper section of $(g_i, g_{i+1})^{r+1+\ell} J$, where $g_i^{r+1+\ell}$ appears as the lowest and uppermost (red) dot, respectively. **Right:** B_{i-1} (dotted green) and A_i (shaded blue) overlap at $g_i^{r+1+\ell}$ resulting in a new staircase, namely that of C_i . Note the required shifts in the x - and y -directions to align B_{i-1} and A_i before summing them up.

Proof. Throughout, we use the notation $\alpha_i := \text{dist}_x(A_i)$ and $\beta_i := \text{dist}_y(B_i)$. By Remark 3.3.15,

$$(r+1)u_i - \alpha_i \geq 0 \quad \text{and} \quad (r+1)v_{i+1} - \beta_{i+1} > 0. \quad (3.3.3)$$

We argue that for $1 \leq i \leq k-1$

$$\text{dist}_x(C_i) = \alpha_{i+1} + u_i(r+1) - \alpha_i \quad (3.3.4)$$

and

$$\text{dist}_y(C_i) = \beta_i + v_{i+1}(r+1) - \beta_{i+1}. \quad (3.3.5)$$

(3.3.4): Both ideals A_{i+1} and B_i are anchored. Hence, the maximal x -degree appearing in the minimal generators of $A_{i+1} \cdot x^{(r+1)u_i - \alpha_i}$ comes from a monomial with y -degree equal to zero and is given by

$$\text{dist}_x A_{i+1} + (r+1)u_i - \alpha_i = \alpha_{i+1} + (r+1)u_i - \alpha_i.$$

Further, the minimal x -degree in the minimal generators of $B_i \cdot y^{(r+1)v_{i+1}}$ is equal to zero. All elements in the summand $B_i \cdot y^{(r+1)v_{i+1}}$ have positive y -degree which implies that $\text{dist}_x(C_i)$ is determined by the maximal x -degree in $A_{i+1} \cdot x^{(r+1)u_i - \alpha_i}$.

(3.3.5): If $(r+1)u_i - \alpha_i > 0$, then we can argue analogously as above. If $(r+1)u_i - \alpha_i = 0$, then Remark 3.3.15 implies that $rv_i = \text{dist}_y J$ and $\beta_i = \text{dist}_y(B_i) = \text{dist}_y J$. Since $\text{dist}_y C_i$ is given by the maximum of the y -degrees of the two summands of C_i , i.e.,

$$\text{dist}_y C_i = \max\{\text{dist}_y B_i + (r+1)v_{i+1} - \text{dist}_y(B_{i+1}), \text{dist}_y A_{i+1}\},$$

and, by Remark 3.3.12, $\text{dist}_y A_{i+1} = (r+1)v_{i+1} + \text{dist}_y J - \beta_{i+1}$, the assertion follows.

We turn our attention to the assertion of the theorem and proceed by induction on k . The basis $k = 1$ is Corollary 3.3.13.

Now let $k > 1$. By the assumption that $(g_1, \dots, g_{k+1})$ is anchored, it follows that $\gcd(g_1, \dots, g_k) = \deg_y g_k = v_k$.

We can apply the induction hypothesis to the anchored ideal $(g_1/y^{v_k}, \dots, g_k/y^{v_k})$ to conclude that

$$\sum_{i=1}^{k-1} \left((g_i, g_{i+1})^{r+1+\ell} J \right) = (L \circledast B_{k-1}) y^{(r+1+\ell)v_k}$$

where

$$L := C_0 \circledast \left(\bigoplus_{i=1}^{k-2} \left(H_i^{\circledast \ell} \circledast C_i \right) \right) \circledast H_{k-1}^{\circledast \ell}.$$

Similarly, setting $u = \deg_x g_k$, we know that

$$(g_k, g_{k+1})^{r+1+\ell} J = (A_k \circledast R) \cdot x^{u(r+1+\ell)} \quad \text{with} \quad R := H_k^{\circledast \ell} \circledast C_k.$$

Therefore,

$$\sum_{i=1}^k \left((g_i, g_{i+1})^{r+1+\ell} J \right) = (L \circledast B_{k-1}) y^{v_k(r+1+\ell)} + (A_k \circledast R) \cdot x^{u(r+1+\ell)},$$

which is equal to

$$\left(L y^{\beta_{k-1}} + B_{k-1} x^{\text{dist}_x L} \right) \cdot y^{v_k(r+1+\ell)} + \left(A_k y^{\text{dist}_y R} + R x^{\alpha_k} \right) \cdot x^{u(r+1+\ell)}. \quad (3.3.6)$$

Before we continue manipulating (3.3.6), we verify a few handy equations. Note that $\text{dist}_x(H_i) = u_i$ and $\text{dist}_y(H_i) = v_i$ for $1 \leq i \leq k$, and recall that $u = \deg_x g_k = \sum_{i=1}^{k-1} u_i$. Thus, using (3.3.4),

$$\begin{aligned} \text{dist}_x L &= \alpha_1 + \ell \sum_{i=1}^{k-1} u_i + \sum_{i=1}^{k-2} \text{dist}_x(C_i) = \alpha_{k-1} + \ell u + (r+1) \sum_{i=1}^{k-2} u_i \\ &= \alpha_{k-1} + (r+1+\ell)u - (r+1)u_{k-1}. \end{aligned}$$

It follows that

$$\text{dist}_x(L \circledast C_{k-1}) = \text{dist}_x L + \text{dist}_x(C_{k-1}) = \alpha_k + (\ell + r + 1)u \quad (3.3.7)$$

and

$$u(r+1+\ell) - \text{dist}_x L = (r+1)u_{k-1} - \alpha_{k-1} \geq 0, \quad (3.3.8)$$

where the last inequality comes from (3.3.3). Moreover, since $\text{dist}_y R = \beta_k + \ell v_k$ we have (using (3.3.3) again)

$$v_k(r+1+\ell) - \text{dist}_y R = (r+1)v_k - \beta_k > 0 \quad (3.3.9)$$

and, using (3.3.5),

$$\text{dist}_y(\mathbf{C}_{k-1} \circledast R) = \text{dist}_y(\mathbf{C}_{k-1}) + \text{dist}_y R = \beta_{k-1} + (\ell + r + 1)v_k. \quad (3.3.10)$$

Using (3.3.8) and (3.3.9), the middle two summands of (3.3.6) can be rearranged to

$$\left(\mathbf{B}_{k-1} y^{(r+1)v_k - \beta_k} + \mathbf{A}_k x^{(r+1)u_{k-1} - \alpha_{k-1}} \right) x^{\text{dist}_x L} y^{\text{dist}_y R}.$$

Thus, using (3.3.7) and (3.3.10), we conclude that (3.3.6) equals

$$L y^{\text{dist}_y(\mathbf{C}_{k-1} \circledast R)} + \mathbf{C}_{k-1} x^{\text{dist}_x L} y^{\text{dist}_y R} + R x^{\text{dist}_x(L \circledast \mathbf{C}_{k-1})} = L \circledast \mathbf{C}_{k-1} \circledast R$$

which completes the proof. $\square$

Remark 3.3.17. Let $h_1, \dots, h_k$ be the link points of

$$S := \sum_{i=1}^k (g_i, g_{i+1})^{r+1} J = \bigcirc_{i=0}^k \mathbf{C}_i.$$

Using the observation of Remark 3.3.10, $\mathbf{C}_i = S : \gcd(h_i, h_{i+1})$, where $h_0 := g_1^{r+1}$ and $h_{k+1} := g_{k+1}^{r+1}$. The link points can be determined by their y -degree, that is,

$$\deg_y h_i = \min \left\{ \deg_y f \mid f \in G(S), \deg_y f \geq (r+1)v_i + r \deg_y g_{i+1} \right\}.$$

In particular, this means that we can determine the minimal generators of $\mathbf{C}_i$ directly from the minimal generators of S , that is,

$$\mathbf{C}_i = \left(g \in G(S) \mid \deg_y h_{i+1} \leq \deg_y g \leq \deg_y h_i \right)_{\mathfrak{J}}.$$

Moreover, also $\mathbf{H}_i$ can be determined from S by choosing its elements according to their y -degree. By Remark 3.3.12 and Theorem 3.3.4,

$$G(\mathbf{H}_i) = \left\{ g \in G \left((g_i, g_{i+1})_{\mathfrak{J}}^{r+1} J \right) \mid rv_i \leq \deg_y g < (r+1)v_i \right\} \uplus \{ x^{\tilde{\alpha}_i - u_i} y^{\tilde{\beta}_i + v_i} \},$$

where $x^{\tilde{\alpha}_i} y^{\tilde{\beta}_i} = \frac{h_i}{\gcd(g_i, g_{i+1})^{r+1}}$. The elements in $\{g \in G((g_i, g_{i+1})^{r+1} J) \mid rv_i \leq \deg_y g < (r+1)v_i\}$ “survive the concatenation” of $\mathbf{A}_i$ and $\mathbf{B}_{i-1}$ to $\mathbf{C}_{i-1}$ as minimal generators (shifted by $x^{(r+1)u_{i-1} - \text{dist}_x(\mathbf{A}_{i-1})}$). It follows that

$$\begin{aligned} \mathbf{H}_i &= \left(\left(g \in G(S) \mid \deg_y h_i \leq \deg_y g < \deg_y g_{i-1}^s \right) + \left(\frac{h_i}{x^{u_i}} y^{v_i} \right) \right)_{\mathfrak{J}} \\ &= \left(g \in G(S) \mid \deg_y h_i \leq \deg_y g < \deg_y g_{i-1}^s \right)_{\mathfrak{J}} + (y^{v_i}). \end{aligned}$$

Corollary 3.3.18. *With the assumptions and notation from Theorem 3.3.16, we have*

$$\mu\left(\sum_{i=1}^k (g_i, g_{i+1})^{r+\ell+1} J\right) = 1 + \sum_{i=0}^k (\mu(C_i) - 1) + \ell \sum_{i=1}^k (\mu(H_i) - 1).$$

Proof. This is an immediate consequence of Theorem 3.3.16 in combination with Remark 3.3.9. $\square$

3.4 Minimal generating sets of powers

We now apply the preceding results to describe the minimal generators of large powers of an ideal I . At the beginning of Section 3.3 we chose to partition the elements of ideals with regular staircase factors based on their y -degrees (Lemma 3.3.2, Remark 3.3.1). Consequently, most definitions and results are phrased with assertions about the y -degree of elements. However, by interchanging the roles of the variable names, the analogous results hold for the x -degrees, too. In this section we take this into account.

Remark 3.4.1. Switching the roles of variables affects the link, cf. Remark 3.3.8, and reverses the order of $g_1, \dots, g_{k+1}$.

Notation 3.4.2. Let I be a monomial ideal and $P = \{g_1, \dots, g_{k+1}\}$ such that $P(I) \subseteq P \subseteq P^*(I)$ and the g_i are ordered in descending y -degree. Further, with δ_P as in Notation 3.2.9, and d_P as in Notation 3.2.13, let $D \geq D_P := (\mu(I) - |P|) \cdot \delta_P + |P| \cdot d_P$. For $\bullet \in \{x, y\}$, we denote

$$r_\bullet(P, D) := \left\lceil D \cdot \max_{1 \leq i \leq k} \frac{\text{dist}_\bullet I}{\text{dist}_\bullet(g_i, g_{i+1})} \right\rceil.$$

Remark 3.4.3. Ordering the elements of P in ascending y -degree instead of ascending x -degree does not affect the values of $r_\bullet(P, D)$.

Definition 3.4.4. Let I be a monomial ideal, fix $\bullet \in \{x, y\}$, and let $P = \{g_1, \dots, g_{k+1}\}$ such that $P(I) \subseteq P \subseteq P^*(I)$ and the g_i are ordered in descending $\bullet$ -degree. Further, let $D \geq D_P$ and $s \geq D + r_\bullet(P, D) + 1$.

We write $r := s - D - 1$ and for $1 \leq i \leq k$, we set $h_i^\bullet$ to be the minimal generator of I^s such that

$$\deg_\bullet h_i^\bullet = \min\{\deg_\bullet f \mid f \in G(I^s), \deg_\bullet f \geq r \text{dist}_\bullet(g_i, g_{i+1}) + (r+1) \deg_\bullet g_{i+1}\},$$

and we set $h_0^\bullet := g_1^s$ and $h_{k+1}^\bullet := g_{k+1}^s$. We define the $(s, \bullet)$ -stable components of I (with respect to P and D) to be

$$C_i := I^s : \gcd(h_i^\bullet, h_{i+1}^\bullet) \text{ for } 0 \leq i \leq k$$

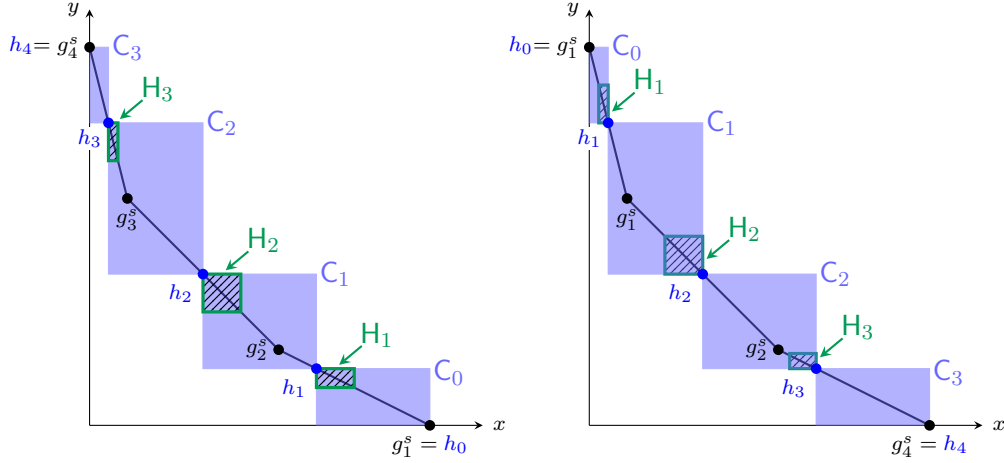


Fig. 3.9: A simplified visualisation of the $(s, \bullet)$ -stable components $(C_i)_{i=0}^k$ and $(H_i)_{i=1}^k$ ($\bullet = x$ on the left and $\bullet = y$ on the right) of an ideal with $k = |P| = 3$.

and for $1 \leq i \leq k$,

$$H_i := (g \in G(I^s) \mid \deg_{\bullet} h_i^{\bullet} \leq \deg_{\bullet} g \leq \deg_{\bullet} g_{i-1}^s)_{\mathfrak{J}} + (\bullet^{\text{dist}_{\bullet}(g_i, g_{i+1})}).$$

If I is not anchored, then we define its $(s, \bullet)$ -stable components to be the $(s, \bullet)$ -stable components of $I_{\mathfrak{J}}$.

Remark 3.4.5. The monomials $h_1^{\bullet}, \dots, h_k^{\bullet}$ in Definition 3.4.4 are by the definition of $(C_i)_{i=0}^k$ the link points of $C_0 \odot \dots \odot C_k$. Note that $h_1^{\bullet}, \dots, h_k^{\bullet}$ are ordered in descending $\bullet$ -degree, see Figure 3.9. Further, for $1 \leq i \leq k$,

$$\deg_{\bullet} g_{i+1}^s \leq \deg_{\bullet} h_i^{\bullet} \leq (r+1) \left(\text{dist}_{\bullet}(g_i, g_{i+1}) + \deg_{\bullet} g_{i+1} \right) = \deg_{\bullet} g_i^{r+1}.$$

The lower bound is due to $r \geq r_{\bullet}(P, D) \geq \frac{D \text{dist}_{\bullet} I}{\text{dist}_{\bullet}(g_i, g_{i+1})}$ and the upper bound follows from Corollary 3.3.6.

We are now set to prove the main theorem of this section, bringing Theorems 3.2.14 and 3.3.16 together.

Theorem 3.4.6. Let $I \subseteq k[x, y]$ be a monomial ideal, $\bullet \in \{x, y\}$, $P(I) \subseteq P \subseteq P^*(I)$, $D \geq D_P$, and $s \geq D + r_{\bullet}(P, D) + 1$. Further, let $(C_i)_{i=0}^k$ and $(H_i)_{i=1}^k$ be the $(s, \bullet)$ -stable components of I with respect to P and D .

Then for all $\ell \geq 0$,

$$I^{s+\ell} = \text{gcd}(I)^{s+\ell} \cdot \left(C_0 \odot \bigodot_{i=1}^k \left(H_i^{\odot \ell} \odot C_i \right) \right).$$

Proof. Without restriction, we assume that $\bullet = y$. As P and D are fixed, we write

$r_y := r_y(P, D)$. Since $I = \gcd(I) \cdot I_{\mathfrak{z}}$, we can carry out the proof under the assumption that I is anchored and multiply by $\gcd(I)^{s+\ell}$ in the end. Let $g_1, \dots, g_{k+1}$ be the elements of P ordered in descending y -degree. We apply Theorem 3.2.14 and obtain that for all $t \geq 0$

$$I^{D+t} = \sum_{i=1}^k (g_i, g_{i+1})^t I^D.$$

Note that the weakly persistent generators of the ideal generated by P are precisely the elements of P . Further, we observe that $\text{dist}_y(I^D) = D \text{dist}_y I$ and hence

$$r := s - D - 1 \geq r_y = \left\lceil \max_{1 \leq i \leq k} \frac{\text{dist}_y(I^D)}{\text{dist}_y(g_i, g_{i+1})} \right\rceil.$$

Therefore, the conditions for Theorem 3.3.16 are satisfied and by applying it, we obtain that for all $\ell \geq 0$

$$I^{s+\ell} = \sum_{i=1}^k (g_i, g_{i+1})^{r+1+\ell} I^D = C'_0 \circledast \bigcirc_{i=1}^k \left(H_i'^{\circledast \ell} \circledast C'_i \right),$$

where H_i' and C'_i are as in Theorem 3.3.16. Observe that the link points of

$$I^s = C'_0 \circledast \dots \circledast C'_k$$

are precisely $h_1^y, \dots, h_k^y$ as in Definition 3.4.4 (cf. Remark 3.3.17). With Remarks 3.3.10 and 3.3.12 this implies that $C'_i = C_i$ and $H_i' = H_i$ are the (s, y) -stable components of I . $\square$

Notation 3.4.7. With the notation of Theorem 3.4.6, let

$$r(P, D) := \min\{r_x(P, D), r_y(P, D)\}.$$

Remark 3.4.8. Theorem 3.4.6 states that all information about large powers of I is encoded in I^s for any $s \geq D + r(P, D) + 1$.

Corollary 3.4.9. For $s \geq D_P + r(P, D_P) + 1$ and $\ell \geq 0$, the computation of $G(I^{s+\ell})$ from $G(I^s)$ takes $\mathcal{O}(\ell)$ additions of (monomial) exponents.

Remark 3.4.10. Theorem 3.4.6 leads to significantly faster computations of large powers of I . For a runtime comparison we refer to Section 3.4.1.

Recall that the Hilbert function of the fibre ring of a monomial ideal—which counts the number of generators of its powers—eventually becomes a polynomial function (cf. [25, Theorem 6.1.3]). Theorem 3.4.6 provides an explicit description of this polynomial.

Corollary 3.4.11. Let $I \subseteq \mathbb{k}[x, y]$ be a monomial ideal, $P(I) \subseteq P \subseteq P^*(I)$, $D \geq D_P$,

and $s \geq D + r_{\bullet}(P, D) + 1$. Then for all $\ell \geq 0$

$$\mu(I^{s+\ell}) = 1 + \sum_{i=0}^k (\mu(C_i) - 1) + \ell \sum_{i=1}^k (\mu(H_i) - 1),$$

where $(C_i)_{i=0}^k, (H_i)_{i=1}^k$ are the $(s, \bullet)$ -stable components of I with respect to P . In particular,

$$\mu(I^{s+\ell}) = \mu(I^s) + \ell (\mu(I^{s+1}) - \mu(I^s)).$$

Remark 3.4.12. For $\bullet \in \{x, y\}$ and $|P| > 2$, we have $r_{\bullet}(P, D_P) \leq D_P \text{dist}_{\bullet}(I)$ and $D_P \leq \mu(I)(\text{dist}_{\bullet} I - 1)$. Thus

$$D_P + r_{\bullet}(P, D_P) + 1 \leq \mu(I) (\text{dist}_{\bullet}(I)^2 - 1) + 1.$$

If $|P| = 2$, then $r(P, D_P) = D_P$ and hence

$$D_P + r(P, D_P) + 1 = 2(\mu(I) - 2) (\min\{\text{dist}_x I, \text{dist}_y I\} - 1) + 1.$$

Corollary 3.4.13. Let $I \subseteq k[x, y]$ be a monomial ideal with $P(I) = \{x^a, y^b\}$, $d := \min\{a, b\}$, and $s \geq 2(\mu(I) - 2)(d - 1) + 1$.

Then for all $\ell \geq 0$

$$I^{s+\ell} = C_0 \odot H_1^{\odot \ell} \odot C_1,$$

where C_0, C_1, H_1 are the $(s, \bullet)$ -stable components of I with respect to $P = P(I)$ and $D = D_P$.

Proof. This is the special case of Theorem 3.4.6 with $k = 1$. □

Remark 3.4.14. Note that in the case $P = \{x^a, y^b\}$, the s -segments of $(x^a, y^b)I^{D_P}$ (Definition 3.3.11) coincide the (s, y) -stable components of I with respect to P , i.e.,

$$A = C_0, \quad H = H_1, \quad \text{and} \quad B = C_1.$$

We now summarize how the minimal generators of I^s and I^{s+1} can be explicitly described from the $(s, \bullet)$ -stable components.

Corollary 3.4.15. Let $I \subseteq k[x, y]$ be an anchored monomial ideal, $\bullet \in \{x, y\}$, $P(I) \subseteq P \subseteq P^*(I)$, where $g_1, \dots, g_{k+1}$ are the elements of P , ordered in descending $\bullet$ -degree. Let $h_0^{\bullet}, \dots, h_{k+1}^{\bullet}$ and $(C_i)_{i=0}^k, (H_i)_{i=1}^k$ be as in Definition 3.4.4, using $D \geq D_P$, and $s \geq D + r_{\bullet}(P, D) + 1$. We set $q_i := \gcd(h_i^{\bullet}, h_{i+1}^{\bullet})$ for $0 \leq i \leq k$ and $\hat{q}_i := \frac{h_i}{f_i}$, where for $1 \leq i \leq k$

$$f_i = \begin{cases} x^{\text{dist}_x(g_i, g_{i+1})} & \text{if } \bullet = y \text{ and} \\ y^{\text{dist}_y(g_i, g_{i+1})} & \text{if } \bullet = x. \end{cases}$$

Then

$$G(I^s) = G(q_0 C_0) \uplus \biguplus_{i=1}^k G^*(q_i C_i),$$

$$G(I^{s+1}) = g_1 \cdot G(q_0 C_0) \uplus \biguplus_{i=1}^k g_{i+1} \left(G^*(q_i C_i) \uplus G^*(\hat{q}_i H_i) \right),$$

and

$$G(I^{s+2}) = g_1^2 G(q_0 C_0) \uplus \biguplus_{i=1}^k g_{i+1}^2 \left(G^*(q_i C_i) \uplus G^*(\hat{q}_i H_i) \right) \uplus \biguplus_{i=1}^k g_i g_{i+1} G^*(\hat{q}_i H_i),$$

where $G^*(J)$ denotes the minimal generating set of a monomial ideal J excluding the minimal generator of maximal $\bullet$ -degree.

Proof. Without restriction, we assume that $\bullet = y$ and write $h_i := h_i^\bullet$. The first assertion follows from Theorem 3.4.6 in combination with Remarks 3.3.10 and 3.4.5. Note that

$$\text{dist}_x(C_0 \uplus \dots \uplus C_{i-1}) = \deg_x h_i \quad \text{and} \quad \text{dist}_y(C_i \uplus \dots \uplus C_k) = \deg_y h_i.$$

Moreover, again by Theorem 3.4.6, we have

$$I^{s+1} = C_0 \uplus H_1 \uplus \dots \uplus H_k \uplus C_k. \quad (3.4.1)$$

Let $\alpha_i := \text{dist}_x H_i = \text{dist}_x(g_i, g_{i+1})$ and $\beta_i := \text{dist}_y H_i = \text{dist}_y(g_i, g_{i+1})$ for $1 \leq i \leq k$. With this notation, $\deg_x g_i = \sum_{j=1}^{i-1} \alpha_j$ and $\deg_y g_i = \sum_{j=i}^k \beta_j$.

For $1 \leq i \leq k$, let w_i be the link point of $(C_0 \uplus H_1 \uplus \dots \uplus H_{i-1} \uplus C_{i-1})$ and $(H_i \uplus C_i \uplus \dots \uplus H_k \uplus C_k)$. Then

$$\deg_x w_i = \deg_x h_i + \sum_{j=1}^{i-1} \alpha_j \quad \text{and} \quad \deg_y w_i = \deg_y h_i + \sum_{j=i}^k \beta_j,$$

which implies $w_i = h_i g_i$.

Similarly, for the link point m_i of $(C_0 \uplus H_1 \uplus \dots \uplus C_{i-1} \uplus H_i)$ and $(C_i \uplus \dots \uplus H_k \uplus C_k)$ we conclude that $m_i = h_i g_{i+1}$.

The monomials $w_1, m_1, w_2, \dots, w_k, m_k$ are the link points of the link (3.4.1) from left to right. With $m_0 := h_0 g_1$, $w_{k+1} := h_{k+1} g_{k+1}$ and Remark 3.3.10, we have

$$I^{s+1} = \sum_{i=0}^k \gcd(m_i, w_{i+1}) C_i + \sum_{i=1}^k \gcd(w_i, m_i) H_i.$$

A straight-forward verification shows that for $1 \leq i \leq k$

$$\gcd(m_i, w_{i+1}) = g_{i+1} q_i \quad \text{and} \quad \gcd(m_i, w_i) = \gcd(g_i, g_{i+1}) h_i = g_{i+1} \hat{q}_i.$$

Finally, it follows from the definition of the link that the individual summands intersect exactly at the linking points which are excluded in $G^*(H_i)$ and $G^*(C_i)$, making them pairwise disjoint. This proves the second assertion.

For the third assertion, we proceed analogously. We determine the link points of

$$\begin{aligned} & (C_0 \otimes H_1^{\otimes 2} \otimes \cdots \otimes H_{i-1}^{\otimes 2} \otimes C_{i-1}) \quad \text{and} \quad (H_i^{\otimes 2} \otimes C_i \otimes \cdots \otimes H_k^{\otimes 2} \otimes C_k) \\ & (C_0 \otimes H_1^{\otimes 2} \otimes \cdots \otimes H_{i-1}^{\otimes 2} \otimes C_{i-1} \otimes H_i) \quad \text{and} \quad (H_i \otimes C_i \otimes \cdots \otimes H_k^{\otimes 2} \otimes C_k) \\ & (C_0 \otimes H_1^{\otimes 2} \otimes \cdots \otimes H_{i-1}^{\otimes 2} \otimes C_{i-1} \otimes H_i^{\otimes 2}) \quad \text{and} \quad (C_i \otimes \cdots \otimes H_k^{\otimes 2} \otimes C_k) \end{aligned}$$

which are $w_i := h_i g_i^2$, $u_i := h_i g_i g_{i+1}$, and $m_i := h_i g_{i+1}^2$. With $m_0 = h_0 g_1^2$ and $w_{k+1} = h_{k+1} g_{k+1}^2$, we have

$$I^{s+2} = \sum_{i=0}^k \gcd(m_i, w_{i+1}) C_i + \sum_{i=1}^k \gcd(w_i, u_i) H_i + \sum_{i=1}^k \gcd(u_i, m_i) H_i.$$

Since $\gcd(m_i, w_{i+1}) = g_{i+1}^2 q_i$, $\gcd(u_i, m_i) = g_{i+1}^2 \hat{q}_i$, and $\gcd(u_i, w_i) = g_i g_{i+1} \hat{q}_i$, the third assertion follows. $\square$

Remark 3.4.16. It follows from the corollary and its proof that $H_i = I^{s+1} : g_{i+1} \hat{q}_i$.

Remark 3.4.17. Corollary 3.4.15 shows how the minimal generators of I^{s+1} and I^{s+2} are computed from $G(I^s)$. For $I^{s+\ell}$ one multiplies the $q_i C_i$ with g_{i+1}^ℓ , and $\hat{H}_i$ with $g_i^{\ell_1} g_{i+1}^{\ell_2}$ for all ℓ_1, ℓ_2 with $\ell_1 + \ell_2 = \ell$.

Corollary 3.4.18. *With the assumptions and notation of Corollary 3.4.15,*

$$G(I^{s+1}) = \biguplus_{f \in G(I^s)} f \cdot G_f,$$

where, with the notation $w_i := \text{dist}_\bullet(g_i, g_{i+1})$,

$$G_f = \begin{cases} \{g_i\} & \text{if } 1 \leq i \leq k, \text{ and } \deg_\bullet h_i^\bullet + w_i < \deg_\bullet f < \deg_\bullet h_{i-1}^\bullet, \\ \{g_i, g_{i+1}\} & \text{if } 1 \leq i \leq k, \text{ and } \deg_\bullet h_i^\bullet \leq \deg_\bullet f \leq \deg_\bullet h_i^\bullet + w_i, \\ \{g_{k+1}\} & \text{if } \deg_\bullet f < \deg_\bullet h_k^\bullet. \end{cases}$$

Proof. Observe that for $1 \leq i \leq k+1$,

$$\begin{aligned} S_i &:= \{f \in G(I^s) \mid \deg_\bullet h_i^\bullet + w_i < \deg_\bullet f < \deg_\bullet h_{i-1}^\bullet\} \\ &\subseteq \{f \in G(I^s) \mid \deg_\bullet h_i^\bullet \leq \deg_\bullet f \leq \deg_\bullet h_{i-1}^\bullet\} \\ &= G(q_{i-1} C_{i-1}). \end{aligned}$$

Moreover, for $1 \leq i \leq k$,

$$\begin{aligned} T_i &:= \{f \in G(I^s) \mid \deg_{\bullet} h_i^{\bullet} \leq \deg_{\bullet} f \leq \deg_{\bullet} h_i^{\bullet} + w_i\} \\ &\subseteq G(q_{i-1}C_{i-1}) \cup \{f \in G(I^s) \mid \deg_{\bullet} h_i^{\bullet} \leq \deg_{\bullet} f \leq \deg_{\bullet} h_i^{\bullet} + w_i\} \\ &= G(q_{i-1}C_{i-1}) \cup G(\hat{q}_i H_i). \end{aligned}$$

The assertion now follows from Corollary 3.4.15. $\square$

Corollary 3.4.19. *With the assumptions and notation of Corollary 3.4.15, let $\ell \in \{1, 2\}$, $g \in G(I^{s+\ell})$ and i such that $\deg_{\bullet} g_{i+1}^{s+\ell} \leq \deg_{\bullet} g \leq \deg_{\bullet} g_i^{s+\ell}$.*

If $\ell = 1$, then

$$g \in \begin{cases} g_i G(I^s) & \text{if } \deg_{\bullet} g \geq \deg_{\bullet} h_i^{\bullet} g_i \\ g_{i+1} G(I^s) & \text{if } \deg_{\bullet} g \leq \deg_{\bullet} h_i^{\bullet} g_i. \end{cases}$$

If $\ell = 2$, then

$$g \in \begin{cases} g_i G(I^{s+1}) & \text{if } \deg_{\bullet} g \geq \deg_{\bullet} h_i^{\bullet} g_i g_{i+1} \\ g_{i+1} G(I^{s+1}) & \text{if } \deg_{\bullet} g \leq \deg_{\bullet} h_i^{\bullet} g_i g_{i+1}. \end{cases}$$

Proof. In the proof of Corollary 3.4.15, we have seen that $h_i^{\bullet} g_i$ is the link point between $g_{i+1}(q_i C_i + \hat{q}_i H_i)$ and $g_i q_i C_i$. Moreover, $\deg_{\bullet} h_i^{\bullet} g_i g_{i+1}$ is the link point between $g_{i+1}(q_i C_i + \hat{q}_i H_i)$ and $g_i g_{i+1} \hat{q}_i H_i + g_i q_i C_i$. The assertion follows from a comparison of degrees. $\square$

We now present examples to conclude this section.

Example 3.4.20. Let $I = (y^2, x^2 y, x^3)$. We apply Corollary 3.4.13 with

$$P = P(I) = \{y^2, x^3\}$$

to give a complete description of the generators of large powers of I .

- (1) We start by computing $D_P = 1$, $r = r_x(P, D_P) = r_y(P, D_P) = 1$, and $s = 3$.
- (2) Next, we compute that $h_1^y = x^6 y^2$ is the minimal generator of I^3 with y -degree at least $r \cdot 2 + 0 = 2$. Recall $h_0^y = y^6$ and $h_2^y = x^9$.
- (3) We compute the $(3, y)$ -stable components with respect to P

$$\begin{aligned} C_0 &= I^3 : y^2 = (y^4, x^2 y^3, x^3 y^2, x^5 y, x^6), \\ H_1 &= I^3 : x^3 y^2 = (y^2, x^2 y, x^3), \text{ and} \\ C_1 &= I^3 : x^6 = (y^2, x^2 y, x^3). \end{aligned}$$

From this we obtain that for all $\ell \geq 0$,

$$I^{3+\ell} = (y^4, x^2y^3, x^3y^2, x^5y, x^6) \oplus (y^2, x^2y, x^3)^{\oplus \ell} \oplus (y^2, x^2y, x^3),$$

and hence

$$G(I^{3+\ell}) = y^{2+2\ell} G(C_0) \uplus \biguplus_{j=1}^{\ell} x^{3+3j} y^{2+2(\ell-j)} G^*(H_1) \uplus x^{6+3\ell} G^*(C_1),$$

where $G^*(J)$ denotes the set of minimal generators of an ideal J , without the minimal generator of largest y -degree. In particular, we have $\mu(I^{3+\ell}) = 7 + 2\ell$.

Example 3.4.21. Let $I = (y^{10}, xy^9, x^2y^5, x^4y^4, x^5y^3, x^6y^2, x^{12}y, x^{15})$. The computations for this example are done in SageMath³. We compute the persistent generators of I

$$P(I) = \{y^{10}, x^2y^5, x^6y^2, x^{15}\},$$

and with $P = P(I)$ we obtain $D_P = 40$, $r(P, D_P) = r_y(D, P) = 200$, and $s = 241$.

With Theorem 3.2.14, we compute

$$I^{241} = \left((y^{10}, x^2y^5)^{201} + (x^2y^5, x^6y^2)^{201} + (x^6y^2, x^{15})^{201} \right) I^{40}.$$

Now, with the notation of Definition 3.4.4,

$$h_0^y = y^{2410}, \quad h_1^y = x^{162}y^{2005}, \quad h_2^y = x^{753}y^{1002}, \quad h_3^y = x^{1815}y^{400}, \quad h_4^y = x^{3615},$$

and hence the $(241, y)$ -stable components of I with respect to P are

$$\begin{aligned} C_0 &= I^{241} : y^{2005}, & H_1 &= I^{241} : x^{160}y^{2005}, \\ C_1 &= I^{241} : x^{162}y^{1002}, & H_2 &= I^{241} : x^{749}y^{1002}, \\ C_2 &= I^{241} : x^{753}y^{400}, & H_3 &= I^{241} : x^{1806}y^{400}, \\ C_3 &= I^{241} : x^{1815}. \end{aligned}$$

Analogously to the example above, these ideals can be used to write down the minimal generators of $I^{241+\ell}$ explicitly. Further, it follows that for all $\ell \geq 0$, $\mu(I^{241+\ell}) = 1688 + 7\ell$.

3.4.1 Runtime in practice

We compare the runtime of our method, implemented in SageMath (Version 9.5), against computations performed in Macaulay2 (Version 1.21; ideals are of type `MonomialIdeal`) All computations were done on a machine equipped with an AMD EPYC 9474F 48-Core Processor @ 4.10GHz (192 cores) and 1536GB RAM.

³`section_5_1.ipynb` as ancillary file on the arXiv page of the paper [48].

We tested the method on four different ideals, which are provided in an additional file³ and choose $P = P(I)$ in all four cases. In the following, we write $D := D_P$ and $s := D + r(P, D) + 1$. We use (s, y) -stable components whenever $r(P, D) = r_y(P, D)$ and (s, x) -stable components otherwise.

The results of the computational comparison are summarized in Table 3.2. The ideal from Example 3.4.21 is I_2 in the table.

	$\mu(I)$	$ P(I) $	$\text{dist } I$
I_1	5	3	7
I_2	8	4	15
I_3	10	7	12
I_4	15	4	24

Table 3.1: An overview of the parameters $\mu(I)$, $|P(I)|$, and $\text{dist } I = \max\{\text{dist}_x I, \text{dist}_y I\}$ of the four test ideals.

Our SageMath implementation begins by computing I^D , after which it applies Theorem 3.2.14 to efficiently compute I^s . Even the initial computation of I^D takes advantage of the fact that all ideals involved are bivariate. For higher powers, we employ Theorem 3.4.6, which offers a substantial speed advantage over the built-in exponentiation of I in Macaulay2. For additional comparison, we have included runtimes for Macaulay2 when using Theorem 3.2.14 to compute $I^{D+\ell}$ from I^D , which already demonstrates significant runtime improvements.

		preprocessing		$s + 10^2$	$s + 10^3$	$s + 10^4$	$s + 10^5$	$s + 10^6$
		I^D	I^s			$I^{s+\ell}$		
I_1		$D = 13$	$s = 45$					
	this method	0.005	0.01	0.04	0.35	4.30	51.35	584.89
	M2 with 3.2.14	0.0006	*	0.05	1.69	1503.34	–	–
	M2 (built-in)	*	*	0.08	22.25	34898.4	–	–
I_2		$D = 40$	$s = 241$					
	this method	0.12	0.30	0.13	0.69	7.44	87.12	980.50
	M2 with 3.2.14	0.02	*	0.36	8.53	3152.86	–	–
	M2 (built-in)	*	*	8.02	411.03	–	–	–
I_3		$D = 76$	$s = 989$					
	this method	0.73	6.29	0.52	1.49	12.12	139.68	1551.92
	M2 with 3.2.14	0.18	*	15.39	69.82	18724.5	–	–
	M2 (built-in)	*	*	607.32	5050.71	–	–	–
I_4		$D = 238$	$s = 2064$					
	this method	28.45	47.13	2.28	4.15	20.53	209.50	2305.38
	M2 with 3.2.14	84.81	*	71.77	176.13	13546.8	–	–
	M2 (built-in)	*	*	> 12h	–	–	–	–

Table 3.2: The two columns under “preprocessing” show the times required to compute I^D and I^s , where “*” indicates that the corresponding method does not use that preprocessing step. The remaining columns present the additional times needed to compute I^{s+10^i} after preprocessing. Cells containing “–” indicate that the estimated computation time would be prohibitively large and is therefore omitted.

4. BUCHBERGER GRAPHS AND THE LCM-COMPLEX¹

In this chapter, we examine two combinatorial structures and the insights they provide into the associated primes of a monomial ideal. We begin with the Buchberger graph, exploring how its subgraphs relate to the associated primes of the underlying monomial ideal. We then introduce the lcm-complex of an ideal and use it to characterize when the maximal ideal is associated to a monomial ideal.

4.1 The Buchberger graph of a monomial ideal

4.1.1 Definition and some properties of the Buchberger graph

In Buchberger's algorithm for computing Gröbner bases, the S-pairs formed from the current set of polynomials are examined, and redundant ones are discarded. An S-pair is a specific combination of two polynomials designed to eliminate their leading terms, which helps detect whether the current set of generators is a Gröbner basis. The minimal set of S-pairs defines a graph on the minimal generators of a monomial ideal. This graph was introduced by Miller and Sturmfels [42] and first appeared under the name "Buchberger graph" in [43].

Definition 4.1.1 (cf. [43, Definition 3.4]). The **Buchberger graph** $\text{Buch}(I)$ of a monomial ideal I with minimal generators $G(I) = \{g_1, \dots, g_s\}$ has vertices $g_1, \dots, g_s$ and an edge (g_i, g_j) whenever $\text{lcm}(g_i, g_j)$ is not properly divisible (Definition 2.1.7) by any minimal generator, that is, if $\text{lcm}(g_i, g_j)$ is a surface monomial of I .

Example 4.1.2. Let $I = (x^2z, x^3y, x^2y^2, xy^3, yz)$. The only pair of generators whose least common multiple is properly divisible by a generator is (x^3y, xy^3) : $\text{lcm}(x^3y, xy^3) = x^3y^3$ is properly divisible by x^2y^2 , c.f. Figure 4.1.

Remark 4.1.3. Let $I \subseteq k[x, y]$ be a monomial ideal in two variables. Then $\text{Buch}(I)$ is a path. To see that, we write $G(I) = \{g_1, \dots, g_s\}$ and $g_i = x^{a_i}y^{b_i}$ for $i \in [s]$. Since $g_1, \dots, g_s$ are minimal generators of I , we can assume that their degrees in x and y are ordered

¹The content of this chapter is the subject of a paper in preparation.

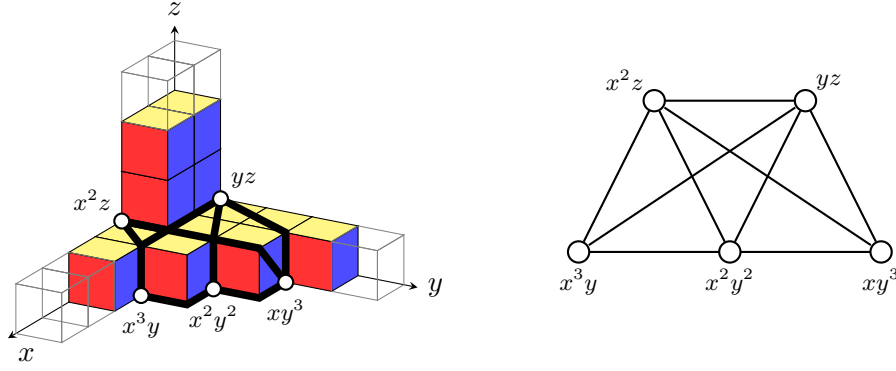


Fig. 4.1: **On the left:** the 3-dimensional staircase surface of the monomial ideal $I = (x^2z, x^2y, x^2y^2, xy^3, yz)$; **on the right:** the Buchberger graph of I .

as follows:

$$\begin{aligned} a_1 &< a_2 < \cdots < a_s, \\ b_1 &> b_2 > \cdots > b_s. \end{aligned}$$

By this ordering it is clear that $g_j \mid_p \text{lcm}(g_i, g_k)$ if and only if $i < j < k$. Therefore, $\text{Buch}(I)$ is a path with edges $\{g_i, g_{i+1}\}$ for all $i \in \{1, \dots, s-1\}$.

Definition 4.1.4 (cf. [43, Definition 3.8]). A monomial ideal $I \subseteq k[x_1, \dots, x_r]$ is called **strongly generic** if every pair of minimal generators g and h satisfies $\deg_i(g) \neq \deg_i(h)$ or $\deg_i(g) = \deg_i(h) = 0$ for every $i \in [r]$.

Remark 4.1.5. If $I \subseteq k[x, y, z]$ is a strongly generic monomial ideal, then $\text{Buch}(I)$ is planar and connected, see [43, Proposition 3.9].

4.1.2 Complete subgraphs of $\text{Buch}(I)$ and $\mathfrak{m}$

In the following, we study the connection of properties of the Buchberger graph of a monomial ideal and its associated primes.

Proposition 4.1.6. Let $I \subseteq k[x_1, \dots, x_r]$ be a monomial ideal. If $\mathfrak{m} \in \text{Ass}(R/I)$, then $\text{Buch}(I)$ has the complete graph K_r as a subgraph.

Proof. Let $w \in \mathbb{N}_0^r$ such that $\mathfrak{m} = I : x^w$ and $m_1, \dots, m_r$ be minimal generators of I with $m_i \mid x^w \cdot x_i$ for $1 \leq i \leq r$. We show that $\{m_i, m_j\}$ is an edge in $\text{Buch}(I)$ for all $i \neq j \in [r]$. For each $1 \leq i \leq r$ we have $m_i \nmid x^w$ and $m_i \mid x^w \cdot x_i$. So, $\deg_i(m_i) = w_i + 1$ and $\deg_\ell(m_i) \leq w_\ell$ for $\ell \neq i$. This implies that the cardinality of the set $\{m_1, \dots, m_r\}$ is

equal to r , and

$$\deg_\ell(\text{lcm}(m_i, m_j)) = \begin{cases} w_\ell + 1, & \ell \in \{i, j\}, \\ \leq w_\ell, & \ell \in [r] \setminus \{i, j\}. \end{cases}$$

Assume that there exists a minimal generator m of I such that $m \mid_p \text{lcm}(m_i, m_j)$. If $\deg_\ell(\text{lcm}(m_i, m_j)) = 0$, then

$$\deg_\ell(m_k) = \deg_\ell(\text{lcm}(m_i, m_j)) = 0 \leq w_\ell.$$

On the other hand, if $\deg_\ell(\text{lcm}(m_i, m_j)) > 0$, then

$$\deg_\ell(m_k) \leq \deg_\ell(\text{lcm}(m_i, m_j)) - 1 \leq w_\ell,$$

hence $m_k \mid x^w$ which is a contradiction. Therefore, the subgraph of $\text{Buch}(I)$ induced by $m_1, \dots, m_r$ is the complete graph K_r . $\square$

Remark 4.1.7. The reverse implication is not true. For example, the Buchberger graph of $I = (xz, xy, yz)$ is a triangle but $\mathfrak{m} \notin \text{Ass}(R/I) = \{(x, y), (x, z), (y, z)\}$, see Figure 4.2.

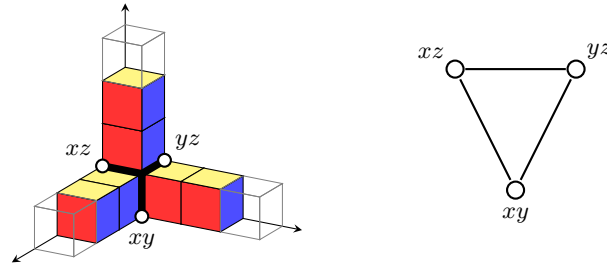


Fig. 4.2: The 3-dimensional staircase surface of the monomial ideal $I = (xy, xz, yz)$ from Remark 4.1.7 is depicted **on the left**. **On the right** is the Buchberger graph of I .

Remark 4.1.8. Note that for any monomial ideal I , the sets of associated primes of height ≥ 2 of the two ideals I and $I : \gcd(I)$ coincide. The Buchberger graphs of I and $I : \gcd(I)$ can differ. For example, the Buchberger graph of $I = (x^3z, xyz, y^3z)$ is a triangle but the Buchberger graph of $I : \gcd(x^3z, xyz, y^3z) = I : z = (x^3, xy, y^3)$ is a path, see Figure 4.3.

Lemma 4.1.9. Let $I \subseteq k[x_1, \dots, x_r]$ be a strongly generic monomial ideal. Then $\mathfrak{m} \in \text{Ass}(R/I)$ if and only if $\text{Buch}(I)$ has the complete graph K_r as a subgraph.

Proof. If $\mathfrak{m} \in \text{Ass}(R/I)$, then $\text{Buch}(I)$ has the complete graph K_r as a subgraph by Proposition 4.1.6. Note that for strongly generic ideals, if we take any r minimal generators $m_1, \dots, m_r$ such that $\text{lcm}(m_1, \dots, m_r)$ is a surface monomial of I , then for all $i \in [r]$ with $\deg_i(\text{lcm}(m_1, \dots, m_r)) > 0$ we have $|\{m_j : \deg_i(m_j) = \deg_i(\text{lcm}(m_1, \dots, m_r))\}| = 1$. That implies that $\deg \text{lcm}(m_1, \dots, m_r) > 0$ in every component, and furthermore $m_1, \dots,$

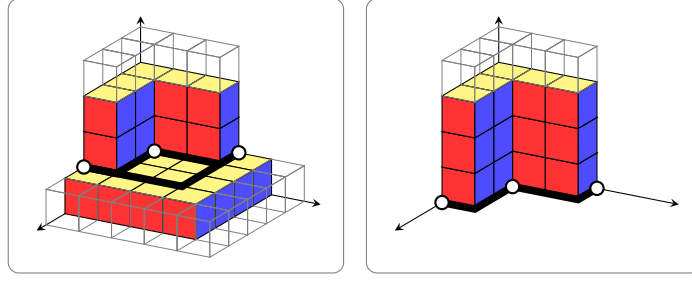


Fig. 4.3: The staircase **on the left** is of the ideal $I = (x^3z, xyz, y^3z)$. There is an edge between the generators x^3z and y^3z in its Buchberger graph since xyz does not properly divide $\text{lcm}(x^3z, y^3z)$. In the ideal $I : z$, the minimal generator xy divides $\text{lcm}(x^3, y^3)$ properly. Therefore, its Buchberger graph is a path. The staircase of $I : z$ is depicted **on the right**.

m_r induce a complete graph K_r in $\text{Buch}(I)$. Also note that in this case,

$$\frac{\text{lcm}(m_1, \dots, m_r)}{x_1 \cdots x_r}$$

is a witness of $\mathfrak{m}$ in I .

Now let $m_1, \dots, m_r$ be generators of I inducing K_r in $\text{Buch}(I)$. If $\text{lcm}(m_1, \dots, m_r)$ is a surface monomial, then we are done. If there exists a generator m of I that properly divides $\text{lcm}(m_1, \dots, m_r)$, then we replace m_1 by m . If $\text{lcm}(m, m_2, \dots, m_r)$ is a surface monomial, then, again, the observation in the beginning completes the proof. Otherwise we repeat this process which has to end after a finite number of steps, since the degree of the least common multiple decreases each time. $\square$

4.2 The lcm-complex of an ideal

As pointed out in Remark 4.1.7, the equivalence

$$\mathfrak{m} \in \text{Ass}(R/I) \iff K_r \text{ is a subgraph of } \text{Buch}(I)$$

does not hold for general monomial ideals. With the motivation to find a similar characterization, we introduce the lcm-complex of an ideal.

4.2.1 Definition and properties of the lcm-complex

Notation 4.2.1. For a set F of monomials, we write $m_F := \text{lcm}(F)$.

Definition 4.2.2. Let I be a monomial ideal. The **lcm-complex** $\mathfrak{L}(I)$ of I is the collection of all subsets $F \subseteq G(I)$ such that

- (1) no minimal generator of I properly divides m_F and
- (2) $m_F \neq m_G$ for all $G \subsetneq F$.

Remark 4.2.3. If we consider the graph that has all 0-dimensional faces of $\mathfrak{L}(I)$ as vertices and all one-dimensional faces as edges, then, by definition, we obtain the Buchberger graph of I .

Example 4.2.4. From left to right, we denote the generators of $I = (x^2z, x^3y, x^2y^2, xy^3, yz)$ from Example 4.1.2 by $g_1, \dots, g_5$. Then $\mathfrak{L}(I)$ consists of

- subsets with one element: $\{g_1\}, \{g_2\}, \{g_3\}, \{g_4\}, \{g_5\}$;
- subsets with two elements: $\{g_1, g_2\}, \{g_1, g_3\}, \{g_1, g_4\}, \{g_1, g_5\}, \{g_2, g_3\}, \{g_2, g_5\}, \{g_3, g_4\}, \{g_3, g_5\}, \{g_4, g_5\}$;
- subsets with three elements: $\{g_1, g_2, g_3\}, \{g_2, g_3, g_5\}, \{g_3, g_4, g_5\}$.

Remark 4.2.5. Let $F \in \mathfrak{L}(I)$. Then by condition (2) in the definition, for each $f \in F$ there exists an $i \in \{1, \dots, r\}$ such that $\deg_i f > \max \{\deg_i g \mid g \in F \setminus \{f\}\}$.

We recall some notions about simplicial complexes:

Definition 4.2.6. A **simplicial complex** Δ on a set $\{1, \dots, n\}$ is a collection of subsets of $\{1, \dots, n\}$ such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. An element $\sigma \in \Delta$ with $|\sigma| = i$ is called an $(i - 1)$ -**dimensional face** of Δ . The **dimension** $\dim(\Delta)$ of Δ is defined as the maximum of the dimensions of its faces. If σ is a maximal face, that is, $\sigma \not\subseteq \tau$ for all $\tau \in \Delta$, then σ is called a **facet** of Δ .

Proposition 4.2.7. Let I be a monomial ideal in $k[x_1, \dots, x_r]$. Then its lcm-complex is a simplicial complex and its dimension is at most $r - 1$.

Proof. Let $F \in \mathfrak{L}(I)$ and $G \subseteq F$. We claim that $G \in \mathfrak{L}(I)$. If some $m \in G(I)$ properly divides m_G , then $m \mid_p m_F$, a contradiction. For the second condition, assume that $H \subsetneq G \subset F$. If $m_H = m_G$, then

$$m_F = \text{lcm}((F \setminus G) \cup G) = \text{lcm}(m_{F \setminus G}, m_G) = \text{lcm}(m_{F \setminus G}, m_H) = \text{lcm}((F \setminus G) \cup H).$$

This is a contradiction since $(F \setminus G) \cup H \subsetneq F$.

For the dimension count, it follows from Remark 4.2.5 that there can be at most r elements in each face. □

Example 4.2.8. The lcm-complex from Example 4.2.4 has dimension 2 and its facets are

$$\{g_1, g_2, g_3\}, \{g_2, g_3, g_5\}, \{g_3, g_4, g_5\}, \{g_1, g_5\}, \{g_1, g_4\}.$$

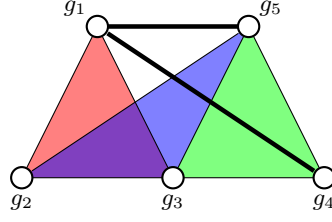


Fig. 4.4: The simplicial complex $\mathfrak{L}(x^2z, x^3y, x^2y^2, xy^3, yz)$ with $g_1 := x^2z$, $g_2 := x^3y$, $g_3 := x^2y^2$, $g_4 := xy^3$, $g_5 := yz$.

Notation 4.2.9. Let $f, g \in k[x_1, \dots, x_r]$ be monomials. Then we denote by $f : g$ the monomial with exponent vector $(\max\{0, \deg_i f - \deg_i g\} : i \in \{1, \dots, r\})$.

Lemma 4.2.10. Let I be a monomial ideal and $f \in R$. Then $\mathfrak{L}(I : f)$ is isomorphic to a simplicial subcomplex of $\mathfrak{L}(I)$.

Proof. Let $\sigma = \{u_1, \dots, u_q\} \in \mathfrak{L}(I : f)$. Then there exist $m_1, \dots, m_q \in G(I)$ such that $u_i = m_i : f$. We show that $\sigma' := \{m_1, \dots, m_q\} \in \mathfrak{L}(I)$. First, we observe that for all $i \in [r]$,

$$\begin{aligned} \deg_i m_\sigma &= \max_{j \in [q]} \{\deg_i u_j\} = \max_{j \in [q]} \{\max\{0, \deg_i m_j - \deg_i f\}\} \\ &= \max\{0, \max_{j \in [q]} \{\deg_i m_j\} - \deg_i f\} \\ &= \max\{0, \deg_i m_{\sigma'} - \deg_i f\} \\ &= \deg_i(m_{\sigma'} : f), \end{aligned}$$

so $m_\sigma = m_{\sigma'} : f$. This observation implies that if there exists a generator $m \in G(I)$ that properly divides $m_{\sigma'}$, then $(m : f) \mid_p (m_{\sigma'} : f) = m_\sigma$. And furthermore, if $\tau' \subsetneq \sigma'$ with $m_{\tau'} = m_{\sigma'}$, then $\tau := \{m : f \mid m \in \tau'\}$ is a proper subset of σ . By the same argument as above, $m_\tau = m_{\tau'} : f = m_{\sigma'} : f = m_\sigma$, a contradiction. $\square$

4.2.2 Connections between $\mathfrak{L}(I)$ and $\text{Ass}(R/I)$

Proposition 4.2.11. Let I be a monomial ideal in $k[x_1, \dots, x_r]$. If I has an associated prime of height n then $\mathfrak{L}(I)$ has a face of dimension $n - 1$.

Proof. Let $P = (x_i \mid i \in S)$ for some set $S \subseteq [r]$ with $|S| = n$. We assume that P is associated to I , i.e., $P = I : w$ for some monomial $w \notin I$. Then $\{x_i \mid i \in S\} \in \mathfrak{L}(P)$ and since $\mathfrak{L}(P) = \mathfrak{L}(I : w)$, Lemma 4.2.10 completes the proof. $\square$

Remark 4.2.12. The reverse implication of Proposition 4.2.11 is in general not true. All primary ideals are counterexamples, for example, if $I = (x, y, z)$, then $\{x, y\} \in \mathfrak{L}(I)$ is a face of dimension one, but $\text{Ass}(R/I) = \{(x, y, z)\}$.

Proposition 4.2.13. *If $\mathfrak{L}(I)$ has a facet of dimension $n - 1$ then I has an associated prime of height n .*

Proof. Let $F = \{m_1, \dots, m_n\}$ be a facet of $\mathfrak{L}(I)$. Then for every $i \in [n]$ there exists an $i_j \in [r]$ such that $\deg_{i_j}(m_i) = \deg_{i_j}(m_F)$ and $\deg_{i_j}(m_k) < \deg_{i_j}(m_i)$ for all $k \in [n] \setminus \{i\}$. Otherwise we could remove m_i from F without changing the least common multiple. Without loss of generality, we can assume that $i_j = i$ for all $i \in [n]$. We show that $w := m_F/x_1 \cdots x_n$ is a witness of $P = (x_1, \dots, x_n)$ in I . By the degree conditions, none of the m_i 's divides w . Also no other generator $m \in G(I) \setminus F$ divides w , since if $m \mid w$ then $m \mid_p m_F$, which contradicts $F \in \mathfrak{L}(I)$. Therefore, we obtain that $w \notin I$. Clearly, $w \cdot x_i \in I$ for all $i \in [n]$, so $P \subseteq I : w$. Assume that $x_k \in I : w$ for some $k > n$, then there exists a generator $g \in G(I)$ such that

$$g \mid \frac{x_k m_F}{x_1 \cdots x_n}, \quad \text{and} \quad g \nmid \frac{m_F}{x_1 \cdots x_n}.$$

Then the degrees of g must fulfill

$$\begin{aligned} \deg_k(g) &= \deg_k(m_F) + 1, \\ \deg_i(g) &\leq \deg_i(m_F) - 1 \text{ for } i \in [n], \text{ and} \\ \deg_i(g) &\leq \deg_i(m_F) \text{ for } i \notin [n] \cup \{k\}. \end{aligned}$$

We show that $F \cup \{g\} \in \mathfrak{L}(I)$. If some generator m divides $m_{F \cup \{g\}}$, then either $m \mid m_F$ but $m \nmid_p m_F$, or $\deg_k(m) = \deg_k(m_{F \cup \{g\}})$. In both cases m does not properly divide $m_{F \cup \{g\}}$. By the conditions on the degrees of the m_i 's and g , we cannot remove any element from $F \cup \{g\}$ and obtain the same least common multiple. So $F \cup \{g\}$ is a face, which is a contradiction to the maximality of F . $\square$

Example 4.2.14. Proposition 4.2.13 implies that the ideal I from Example 4.2.4 has at least one associated prime of height two, and $(x, y, z) \in \text{Ass}(R/I)$.

The reverse implication of Proposition 4.2.13 is not true: The ideal (x^2y, xy^2, z) has associated primes $\{(x, y, z), (x, z), (y, z)\}$. In particular, there are associated primes of height two. However, its lcm-complex consists of $\{x^2y, xy^2, z\}$ and its subsets, so it has no facet of dimension one. Despite that, we can characterize when the maximal ideal is associated in terms of the facets of I :

Corollary 4.2.15. *Let I be a monomial ideal in $k[x_1, \dots, x_r]$. Then $\mathfrak{m} \in \text{Ass}(R/I)$ if and only if $\dim \mathfrak{L}(I) = r - 1$.*

Proof. This immediately follows from Propositions 4.2.11 and 4.2.13. $\square$

Corollary 4.2.16. *Let I be a monomial ideal in $k[x_1, \dots, x_r]$ and f a monomial. If $\mathfrak{m} \in \text{Ass}(R/I : f)$ then also $\mathfrak{m} \in \text{Ass}(R/I)$.*

Proof. If $\mathfrak{m} \in \text{Ass}(R/I : f)$, Proposition 4.2.11 implies that $\mathfrak{L}(I : f)$ has a face of dimension $r - 1$. By Lemma 4.2.10 also $\mathfrak{L}(I)$ has a face of dimension $r - 1$, which must be a facet. Consequently $\mathfrak{m} \in \text{Ass}(R/I)$. $\square$

5. ASSOCIATED PRIMES OF POWERS OF MONOMIAL IDEALS IN THREE VARIABLES¹

Let I be a monomial ideal in $k[x, y, z]$. To study the associated primes of powers of I , we draw on the results from Chapter 3 about the structure of monomial ideals in two variables. For a monomial ideal in any number of variables, we define the **pattern-stability number** of I as

$$\mu(I) \left((\text{dist } I)^2 - 1 \right) + 1,$$

where $\text{dist } I$ denotes the maximum exponent of any variable appearing in the minimal generators of I . If I is a monomial ideal in $k[x, y, z]$, then $\text{dist } I = \max\{\text{dist}_x I, \text{dist}_y I, \text{dist}_z I\}$, cf. Definition 3.1.4.

From Chapter 3, we recall that after this number, the staircases of powers of a bivariate monomial ideal follow a regular, predictable pattern, described explicitly in Theorem 3.4.6. This structural behavior of bivariate monomial ideals will be a key ingredient for arguments of this chapter, where we relate the pattern-stability number to the stability index of monomial ideals in three variables.

Recall from Fact 2.3.19 that the set of minimal primes of a monomial ideal remains invariant under taking powers; that is, for any monomial ideal I , we have $\text{Min}(R/I^n) = \text{Min}(R/I)$ for all $n \geq 0$. By Corollary 2.1.49 and Remark 2.1.33, it suffices to consider ideals whose minimal primes all have height at least two. If (x, y, z) is a minimal prime of I , then it is the unique associated prime of I , and consequently, I and all of its powers are (x, y, z) -primary.

Therefore, it remains to consider the case in which all minimal primes of I have height exactly two. In this setting, the only possible embedded associated prime is the maximal monomial ideal (x, y, z) . It follows that the stability index $\text{stab}(I) \leq s$ if either

$$(x, y, z) \in \text{Ass}(R/I^n) \text{ for all } n \geq s,$$

or

$$(x, y, z) \notin \text{Ass}(R/I^n) \text{ for all } n \geq s.$$

We give a bound for $\text{stab}(I)$ in terms of the pattern-stability numbers of certain related bivariate monomial ideals in the two cases $\text{Min}(R/I) = \{(x, y)\}$ (Theorem 5.2.11) and $\text{Min}(R/I) = \{(x, y), (x, z)\}$ (Theorem 5.2.20). We suspect that similar techniques can be applied to the case $\text{Min}(R/I) = \{(x, y), (x, z), (y, z)\}$.

¹The content of this chapter is the subject of a paper in preparation jointly with Roswitha Rissner.

5.1 Preliminaries

We begin by establishing preliminary properties that will be used throughout the remainder of this chapter. The following lemma is stated in the general setting of monomial ideals in r variables, although our primary applications will concern the case $r = 3$.

Notation 5.1.1. For a monomial $f = x_1^{a_1} \cdots x_r^{a_r}$ in $\mathbf{k}[x_1, \dots, x_r]$ and $i \in [r]$, we denote by f_{x_i} the monomial that is obtained by setting the exponent of x_i to zero, that is,

$$f_{x_i} := x_1^{a_1} \cdots x_{i-1}^{a_{i-1}} x_{i+1}^{a_{i+1}} \cdots x_r^{a_r}.$$

Definition 5.1.2. For a monomial ideal $I \subseteq \mathbf{k}[x_1, \dots, x_r]$ and $1 \leq i \leq r$, we denote by I_{x_i} the saturation of I with respect to x_i , that is, $I_{x_i} = I : x_i^\infty$.

Lemma 5.1.3. Let I be a monomial ideal in $\mathbf{k}[x_1, \dots, x_r]$, and let $1 \leq i \leq r$. Then for every $n \in \mathbb{N}$, we have

$$(I^n)_{x_i} = (I_{x_i})^n.$$

Proof. For an easier notation, we write x instead of x_i . In order to prove that the ideal on the right is included in the ideal on the left, take $f_1, \dots, f_n \in I : x^\infty$, that is, there exists an $N \in \mathbb{N}$ large enough such that for all $i \in [n]$ we have $f_i x^N \in I$. This implies that $\prod_{i=1}^n f_i x^N = f_1 \cdots f_n x^{nN} \in I^n$, and hence $f_1 \cdots f_n \in I^n : x^\infty$. For the reverse inclusion, take $f \in I^n : x^\infty$, that is, there exists an $N \in \mathbb{N}$ such that $f x^N \in I^n$. Then we can write $f x^N = \prod_{i=1}^n g_i$, where $g_i \in I$. We have

$$(f x^N)_x = \left(\prod_{i=1}^n g_i \right)_x = \prod_{i=1}^n (g_i)_x \in (I : x^\infty)^n.$$

Since $(f x^N)_x = f_x \mid f$, this finishes the proof. $\square$

5.2 Upper bounds for the stability index

Recall from Chapter 3, that we denote the set of persistent generators of a monomial ideal I in $\mathbf{k}[x, y]$ by $P(I)$ (Definition 3.2.1), and the set of weakly persistent generators by $P^*(I)$ (Definition 3.2.7). By Remark 3.4.12, for any set of monomials P satisfying $P(I) \subseteq P \subseteq P^*(I)$, the pattern-stability number of I can be bounded from below by

$$\mu(I) \left((\text{dist } I)^2 - 1 \right) + 1 \geq D_P + \min\{r_x(P, D_P), r_y(P, D_P)\} + 1, \quad (5.2.1)$$

where $r_x(P, D_P)$, $r_y(P, D_P)$, and D_P are as in Notation 3.4.2. In particular, this bound guarantees that Theorem 3.4.6 applies to all powers greater than or equal to the pattern-stability number.

Remark 5.2.1. Let I be a monomial ideal in $k[x, y, z]$. Then

$$\text{dist}(I_\bullet) \leq \text{dist } I, \quad \text{and} \quad \mu(I_\bullet) \leq \mu(I)$$

holds for $\bullet \in \{x, y, z\}$. Thus, the pattern-stability number of I is an upper bound for the pattern-stability numbers of I_x , I_y , and I_z .

We extend the notion of weakly persistent generators of bivariate monomial ideals to monomial ideals in any number of variables, that is, if $I \subseteq k[x_1, \dots, x_r]$, then

$$P^*(I) := \{f \in G(I) \mid f^n \in G(I^n) \text{ for all } n \in \mathbb{N}\}.$$

The weakly persistent generators of an ideal can be characterized geometrically using its Newton polyhedron $\mathcal{C}(I)$ (Definition 2.2.15):

Lemma 5.2.2. *Let I be a monomial ideal in $k[x_1, \dots, x_r]$, and let $x^a \in I$ for some $a \in \mathbb{N}_0^r$. Then $x^a \in P^*(I)$ if and only if a is in the topological boundary of the Newton polyhedron of I , i.e., $x^a \in \partial\mathcal{C}(I)$.*

Proof. Let $G(I) = \{x^{a_1}, \dots, x^{a_s}\}$ for $a_1, \dots, a_s \in \mathbb{N}_0^r$. If $a \in \mathbb{N}_0^r$ such that $x^a \in I$ and $a \notin \partial\mathcal{C}(I)$, then there exists $c \in \partial\mathcal{C}(I) \cap \mathbb{Q}^r$ such that $c \leq a$ and $c \neq a$. We can write

$$c = \alpha_1 a_1 + \dots + \alpha_s a_s$$

for $\alpha_1, \dots, \alpha_s \in \mathbb{Q} \cap [0, 1]$ with $\alpha_1 + \dots + \alpha_s = 1$. Let $d \in \mathbb{N}$ be the common denominator of $\alpha_1, \dots, \alpha_s$. Then $x^{dc} \in I^d$, and further $x^{dc} \mid x^{da}$ and $x^{dc} \neq x^{da}$. Thus, this implies that $x^a \notin P^*(I)$.

On the other hand, if $x^a \in G(I) \setminus P^*(I)$, then there exists an $n \in \mathbb{N}$ such that $x^{na} \notin G(I^n)$. That is, there exist $\beta_1, \dots, \beta_s \in \mathbb{N}_0$ such that $\beta_1 + \dots + \beta_s = n$,

$$\sum_{i=1}^s \beta_i a_i \leq na, \quad \text{and} \quad \sum_{i=1}^s \beta_i a_i \neq na.$$

Dividing by n yields

$$\sum_{i=1}^s \frac{\beta_i}{n} a_i \leq a, \quad \text{and} \quad \sum_{i=1}^s \frac{\beta_i}{n} a_i \neq a.$$

Since $\sum_{i=1}^s \frac{\beta_i}{n} a_i \in \mathcal{C}(I)$, this implies that $a \notin \partial\mathcal{C}(I)$. □

Remark 5.2.3. By Lemma 5.2.2, every weakly persistent generator of an ideal I is an element of a face of $\mathcal{C}(I)$. Since the preimage of a face of a polytope under any projection is again a face, cf. [70, Lemma 7.10], it follows that if $f \in G(I)$ satisfies $f_\bullet \in P^*(I_\bullet)$, then f must belong to $P^*(I)$.

From now on, if not explicitly stated otherwise, all ideals are in three variables x , y and z .

5.2.1 One minimal prime: $\text{Min}(R/I) = \{(x, y)\}$

We begin by examining ideals whose minimal prime ideals consist of a single prime of height two. We assume that $\text{Min}(R/I) = \{(x, y)\}$. The choice of (x, y) as the only minimal prime of height two is arbitrary; analogous results hold for any other such prime by permuting the variables.

Remark 5.2.4. Let I be a monomial ideal in $k[x, y, z]$ with $\text{Min}(R/I) = \{(x, y)\}$.

- (1) By Remark 2.1.35(2) no power of z appears as a minimal generator of I .
- (2) Again by Remark 2.1.35(2), since (x) , (z) and (x, z) are not associated primes of I , the staircase of I must be bounded in y -direction. Analogously, it is also bounded in x -direction. Thus, there exist generators x^a and y^b for some $a, b \in \mathbb{N}$.
- (3) By the minimality of (x, y) it follows with Fact 2.1.31 and Remark 2.1.32 that I_z is the uniquely determined (x, y) -primary component of I . Specifically, this implies that $\mathfrak{m} \in \text{Ass}(R/I)$ if and only if $I \neq I_z$.

Notation 5.2.5. For a monomial ideal $I \subseteq k[x, y, z]$, we denote

$$I_0 := (f \in G(I) : z \nmid f) \subseteq k[x, y].$$

Remark 5.2.6. Observe that $\text{dist } I_0 \leq \text{dist } I$, and $\mu(I_0) \leq \mu(I)$. Thus, the pattern-stability number of I_0 is at most the pattern-stability number of I .

Lemma 5.2.7. Let I be a monomial ideal in $k[x, y, z]$ with $\text{Min}(R/I) = \{(x, y)\}$. Then $(x, y, z) \in \text{Ass}(R/I^n)$ if and only if $I_0^n \neq I_z^n$.

Proof. Note that $I_0^n = (I^n)_0$ and, by Lemma 5.1.3, we have $I_z^n = I^n : z^\infty$. Therefore, it suffices to prove the claim for the case $n = 1$. The inclusions $I \subseteq I_z$ and $I_0 \subseteq I$ always hold. Therefore, $I_z = I_0$ implies $I = I_0 = I_z$. By Remark 5.2.4(3), we conclude that $(x, y, z) \notin \text{Ass}(R/I)$.

Conversely, suppose $(x, y, z) \notin \text{Ass}(R/I)$. Then, by Remark 5.2.4(3), it follows that $I = I_z$, which implies $\text{supp}(I) = \{x, y\}$, and therefore $I = I_0$. $\square$

Lemma 5.2.8. If $I_0^n = I^n$ for some $n \in \mathbb{N}$, then also $I_0^{n+1} = I^{n+1}$.

Proof. Let $f_1 \cdots f_{n+1} \in G(I^{n+1})$. Then for every $i \in \{1, \dots, n+1\}$ the product

$$g_i := \prod_{j \neq i} f_j$$

is a minimal generator of I^n . Since no minimal generator of I^n is divisible by z , it follows that $z \nmid g_i$ for all $i \in \{1, \dots, n+1\}$. Since $z \nmid g_1$ it follows that $z \nmid f_j$ for all $j > 1$, and since $z \nmid g_2$ it also follows that $z \nmid f_1$. Hence $z \nmid f_1 \cdots f_{n+1}$. $\square$

Lemma 5.2.9. *If $\text{Min}(R/I) = \{(x, y)\}$ and $(x, y, z) \notin \text{Ass}(R/I^n)$ for some $n \in \mathbb{N}$, then $(x, y, z) \notin \text{Ass}(R/I^N)$ for all $N \geq n$.*

Proof. By Lemma 5.2.7 we know that $(x, y, z) \notin \text{Ass}(R/I^n)$ if and only if I^n has no z in any of its minimal generators and by Lemma 5.2.8 the assertion follows. $\square$

Proposition 5.2.10. *If $\text{Min}(R/I) = \{(x, y)\}$ and $P^*(I_0) \neq P^*(I_z)$, then $\mathfrak{m} \in \text{Ass}(R/I^n)$ for all $n \in \mathbb{N}$ and hence, $\text{stab}(I) = 1$.*

Proof. Note that $I_0 \subseteq I_z$ and also $P^*(I_z) \cap I_0 \subseteq P^*(I_0)$. By the assumption that $P^*(I_0) \neq P^*(I_z)$, there exists a $g \in P^*(I_z) \setminus I_0$ and hence $g^n \in G(I_z^n) \setminus I_0^n$ for all $n \in \mathbb{N}$. The assertion follows from Lemma 5.2.7. $\square$

We are now set to prove the main result of this section.

Theorem 5.2.11. *Let I be a monomial ideal in $k[x, y, z]$ such that $\text{Min}(R/I) = \{(x, y)\}$. Then the pattern-stability number of I is an upper bound for the stability index of I , i.e.,*

$$\text{stab}(I) \leq \mu(I)((\text{dist } I)^2 - 1) + 1.$$

Proof. In Proposition 5.2.10 we established that $\text{stab}(I) = 1$ in the case that $P^*(I_0) \neq P^*(I_z)$. We now consider the case that $P^*(I_0) = P^*(I_z)$. By Lemma 5.2.9 it is left to show that if (x, y, z) is associated to a power $s \geq \mu(I)((\text{dist } I)^2 - 1) + 1$, then (x, y, z) is also associated to all higher powers. So we assume that $(x, y, z) \in \text{Ass}(R/I^s)$, which is by Lemma 5.2.7 equivalent to $I_0^s \neq I_z^s$, and show that

$$I_0^{s+1} \neq I_z^{s+1}$$

holds, which then proves the claim.

By Remarks 5.2.1 and 5.2.6 the pattern-stability numbers of both I_z and I_0 are bounded above by s . Therefore, due to the inequality (5.2.1), we can apply Theorem 3.4.6 to I_0 and I_z , using $\bullet = y$, $P = P^*(I_z) = P^*(I_0)$, and $D = (\mu(I) - |P|) \cdot \delta_P + |P| \cdot d_P$ for both I_0 and I_z . With the notation from this theorem there exist C_i and C'_i such that

$$I_0^s = \gcd(I_0)^s \cdot (C_0 \circledast \cdots \circledast C_k) \tag{5.2.2}$$

and

$$I_z^s = \gcd(I_z)^s \cdot (C'_0 \circledast \cdots \circledast C'_k), \tag{5.2.3}$$

where $k + 1 = |P|$. Note that under the assumption $\text{Min}(R/I) = \{(x, y)\}$, we have $\gcd(I_0) = \gcd(I_z) = 1$ (see Remark 5.2.4(2)). Let $h_1, \dots, h_k$ denote the link points in (5.2.2) and let $h'_1, \dots, h'_k$ denote the link points in (5.2.3).

Case 1: If $h_i = h'_i$ holds for all $i \in [k]$, then by the assumption that $I_0^s \neq I_z^s$, there must exist an $i \in \{1, \dots, k\}$ such that $C_i \neq C'_i$. In this case, Theorem 3.4.6 implies that $I_0^{s+\ell} \neq I_z^{s+\ell}$ for all $\ell \geq 0$.

Case 2: We now consider the case that there exists an $i \in [k]$ such that $h_i \neq h'_i$. Let $g_1, \dots, g_{k+1}$ be the elements of P ordered in descending y -degree. We recall from Definition 3.4.4 that with $\bullet = y$, $D = D_P$, and $r := s - D - 1$ and $d := r \text{dist}_y(g_i, g_{i+1}) + (r + 1) \deg_y g_{i+1}$, the monomial h_i is defined to be the minimal generator of I_0^s such that

$$\deg_y h_i = \min\{\deg_y f \mid f \in G(I_0^s), \deg_y f \geq d\},$$

and h'_i is defined to be the minimal generator of I_z^s such that

$$\deg_y h'_i = \min\{\deg_y f \mid f \in G(I_z^s), \deg_y f \geq d\}.$$

Thus, under the assumption that h_i and h'_i are not equal, it must hold that either $h_i \notin G(I_z^s)$ or $h'_i \notin G(I_0^s)$. If $h_i \notin G(I_z^s)$, then necessarily $g_{i+1}h_i \notin G(I_z^{s+1})$. However, by Corollary 3.4.18, we have $g_{i+1} \cdot h_i \in G(I_0^{s+1})$, and therefore $I_0^{s+1} \neq I_z^{s+1}$. The analogous argument holds if $h'_i \notin G(I_0^s)$. $\square$

5.2.2 Two minimal primes: $\text{Min}(R/I) = \{(x, y), (x, z)\}$

We now turn to the case where the set of minimal primes of I consists of two primes of height two. Without loss of generality, we assume that

$$\text{Min}(R/I) = \{(x, y), (x, z)\}.$$

Equivalent results hold for all other configurations involving two minimal primes of height two and are covered by permuting the variables.

Remark 5.2.12. Let I be a monomial ideal with $\text{Min}(R/I) = \{(x, y), (x, z)\}$. Then we recall from Remark 2.1.35(2) that the following hold:

- There exists $m \in \mathbb{N}$ such that $x^m \in G(I)$.
- No power of y or z lies in I , that is, neither $y^k \in I$ nor $z^\ell \in I$ for any $k, \ell \in \mathbb{N}$.

Furthermore, by Fact 2.1.31 and Remark 2.1.32, I_z is the unique (x, y) -primary component of I and I_y is the unique (x, z) -primary component of I . In particular,

$$\mathfrak{m} \in \text{Ass}(R/I) \iff I \neq I_z \cap I_y.$$

We present a generalized version of Lemma 5.2.7:

Lemma 5.2.13. *Let I be a monomial ideal in $k[x, y, z]$, and $j \in \mathbb{N}$. Write $d_{\leq j} := \gcd\{g \in G(I) : \deg_z g \leq j\}$. If there exists a minimal generator $g \in G(I)$ such that $\deg_z g > j$ and $d_{\leq j} \mid g$, then $\mathfrak{m} \in \text{Ass}(R/I)$.*

Proof. We write $d := \text{lcm}(d_{\leq j}, z^j)$ and $g : d$ as in Notation 4.2.9. Then from $d_{\leq j} \mid g$, it follows that $g : d$ is a minimal generator of $I : d$, and since $\deg_z g > j$, the z -degree of $g : d$ is positive. Further, we have $\text{Min}(I : d) = \{(x, y)\}$, so we can apply Lemma 5.2.7 to obtain that $\mathfrak{m} \in \text{Ass}(R/I : d)$ and by Corollary 4.2.16 also $\mathfrak{m} \in \text{Ass}(R/I)$. $\square$

Proposition 5.2.14. *Let I be a monomial ideal with $\text{Min}(R/I) = \{(x, y), (x, z)\}$. If*

$$\{f \in G(I) \mid f_z \in P^*(I_z)\} \neq \{f \in G(I) \mid f_y \in P^*(I_y)\},$$

then $\mathfrak{m} \in \text{Ass}(R/I^n)$ for all $n \geq \text{dist}_x I$ and hence $\text{stab}(I) \leq \text{dist}_x I$.

Proof. Without loss of generality we can assume that there exists an $f \in G(I)$ such that $f_z \in P^*(I_z)$ and $f_y \notin P^*(I_y)$. Recall that $f_z \in P^*(I_z)$ implies that $f \in P^*(I)$ (Remark 5.2.3). Since $f_y \notin P^*(I)$, there exist an $n \in \mathbb{N}$ and $H \in I^n$ such that $H_y \mid f_y^n$ and $H_y \neq f_y^n$. Thus,

$$(1) \deg_z H \leq \deg_z f^n,$$

$$(2) \deg_x H \leq \deg_x f^n,$$

and, since $f^n \in G(I^n)$ and at least one of the inequalities (1) and (2) must be strict, also

$$(3) \deg_y H > \deg_y f^n.$$

We show that we can choose H such that $\deg_z H < \deg_z f^n$:

Since $f_y \notin P^*(I_y)$, it follows that also $f_y \notin P(I_y)$ and hence there exist $g, h \in G(I) \setminus \{f\}$ such that f_y is in the integral closure $\overline{(g_y, h_y)}$. Therefore, f_y lies between g_y and h_y , and we can assume that $\deg_x g < \deg_x f < \deg_x h$, and consequently $\deg_z g > \deg_z f > \deg_z h$. Furthermore, by Proposition 3.2.3, we obtain the divisibility relation $g_y^\alpha h_y^{n-\alpha} \mid f_y^n$, where $n = \text{dist}_x(g, h)$ and $\alpha = \text{dist}_x(f, h)$. As outlined in the proof of Proposition 3.2.3, the degrees fulfill

- $\deg_z g^\alpha h^{n-\alpha} < \deg_z f^n$ and
- $\deg_x g^\alpha h^{n-\alpha} = \deg_x f^n$.

We choose $H = g^\alpha h^{n-\alpha} \in I^n$.

Since $\text{Min}(R/I) = \{(x, y), (x, z)\}$, we recall from Remark 5.2.12 that there exists an $m \in \mathbb{N}$ such that x^m is a minimal generator of I . Then $m > \deg_x f$ must hold. Write $j := \deg_z H$. Then $\deg_z f^n > j$ and

$$\gcd\{g \in G(I^n) : \deg_z g \leq j\} \mid \gcd(H, x^{mn}) = x^{\deg_x H} \mid f^n.$$

It therefore follows that $\mathfrak{m} \in \text{Ass}(R/I^n)$ by Lemma 5.2.13. We can lift the same argument to higher powers of I , since

$$\gcd(Hf^\ell, x^{(m+\ell)n}) = x^{\deg_x H + \ell \deg_x f} \mid f^{n+\ell}$$

and $\deg_z(f^{n+\ell}) > j + \ell \deg_z f = \deg_z(Hf^\ell)$, so we can again apply Lemma 5.2.13. $\square$

It remains to consider the case $\{f \in G(I) \mid f_z \in P^*(I_z)\} = \{f \in G(I) \mid f_y \in P^*(I_y)\}$. We develop a number of preliminary results.

Proposition 5.2.15. *Let I be a monomial ideal in $k[x, y, z]$.*

- *For every $g \in I_z$ denote h_g the minimal generator of I_y with x -degree equal to $\max\{\deg_x h \mid h \in G(I_y), \deg_x h \leq \deg_x g\}$.*
- *For every $h \in I_y$ denote g_h the minimal generator of I_z with x -degree equal to $\max\{\deg_x g \mid g \in G(I_z), \deg_x g \leq \deg_x h\}$.*

Then $G(I_z \cap I_y) = \{\text{lcm}(g, h_g) \mid g \in G(I_z)\} \cup \{\text{lcm}(g_h, h) \mid h \in G(I_y)\}$.

Proof. Let $g \in G(I_z)$. Then

$$\text{lcm}(g, h_g) = x^{\deg_x g} y^{\deg_y g} z^{\deg_z h_g}.$$

For every $h \in G(I_y)$ with $\deg_x h < \deg_x h_g$ it follows that $\deg_z h > \deg_z h_g$ and therefore

$$\text{lcm}(g, h_g) \mid \text{lcm}(g, h) = x^{\deg_x g} y^{\deg_y g} z^{\deg_z h}.$$

To show that $\text{lcm}(g, h_g) \in G(I_z \cap I_y)$, we assume that

$$\text{lcm}(g', h') \mid \text{lcm}(g, h_g) \tag{5.2.4}$$

for some $g' \in G(I_z)$ and $h' \in G(I_y)$. If the x -degree of g' is less than the x -degree of g , then $\deg_y g' > \deg_y g$ must follow and hence the divisibility relation (5.2.4) is not fulfilled. Since we also cannot choose g' with a larger x -degree than g , it must follow that $g' = g$.

We already excluded the case where $\deg_x h' < \deg_x h_g$. However, if $\deg_x h' > \deg_x h_g$, then by the choice of h_g , it follows that $\deg_x h' > \deg_x g$ and therefore

$$\deg_x(\text{lcm}(g, h_g)) < \deg_x(\text{lcm}(g, h')).$$

Again, the divisibility relation (5.2.4) is not fulfilled. The assertions for $\text{lcm}(g_h, h)$ are proven analogously. $\square$

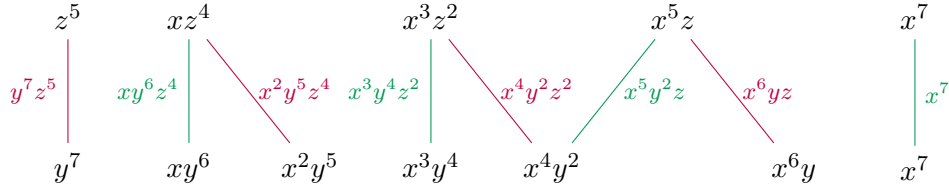


Fig. 5.1: An illustration of an example of how to apply Proposition 5.2.15: We consider an ideal with $I_z = (y^7, xy^6, x^2y^5, x^3y^4, x^4y^2, x^6y, x^7)$, whose generators are ordered in increasing x -degree at the bottom, and $I_y = (z^5, xz^4, x^3z^2, x^5z, x^7)$ with generators listed at the top of the figure. The intersection $I_z \cap I_y$ is minimally generated by the labels of the edges connecting the top row with the bottom row. The edges of this bipartite graph are precisely $\{g, h_g\}$ for $g \in G(I_z)$ and $\{g_h, h\}$ for $h \in G(I_y)$, with h_g and g_h as defined in Proposition 5.2.15.

Corollary 5.2.16. *If $\text{Min}(I) = \{(x, y), (x, z)\}$ and there exist $f_1, \dots, f_s \in G(I)$ such that*

$$G(I_z) = \{(f_1)_z, \dots, (f_s)_z\} \text{ and}$$

$$G(I_y) = \{(f_1)_y, \dots, (f_s)_y\},$$

then $I = I_z \cap I_y$. In particular, $\mathfrak{m} \notin \text{Ass}(R/I)$.

Proof. Note that $I \subseteq I_z \cap I_y$ always holds. Due to Proposition 5.2.15, we have

$$I_z \cap I_y = \left(\text{lcm}((f_i)_z, (f_i)_y) \mid 1 \leq i \leq s \right).$$

Therefore, $I_z \cap I_y = (f_1, \dots, f_s)$ which is a subset of I . As mentioned in Remark 5.2.12, it now follows that $\mathfrak{m} \notin \text{Ass}(R/I)$. $\square$

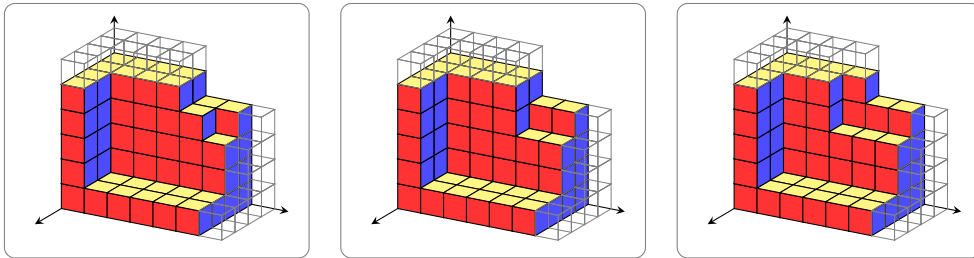


Fig. 5.2: From left to right are the staircases of $I = (x^4, x^2yz, y^4z^4, xy^kz^3)$, where $k = 5, 4, 3$. In all three cases, the sets $\{f \in G(I) \mid f_z \in P^*(I_z)\}$ and $\{f \in G(I) \mid f_y \in P^*(I_y)\}$ coincide and are equal to $\{x^4, x^2yz, y^4z^4\}$. In the ideals **on the left** and **in the middle**, the corresponding projection of the generator xy^kz^3 is in $G(I_y)$ but not in $G(I_z)$. The ideal **on the right** fulfills the requirements of Corollary 5.2.16.

Remark 5.2.17. Let $f = \text{lcm}(f_1, f_2)$ for $f_1 = x^a y^b \in I_z$ and $f_2 = x^{a'} z^c \in I_y$. Then $f = x^{\max\{a, a'\}} y^b z^c$, and therefore, if $a \geq a'$, then $f_z = f_1$ and if $a \leq a'$, then $f_y = f_2$.

Lemma 5.2.18. *Let I be a monomial ideal in $k[x, y, z]$. Then $G(I_z \cap I_y) \cap I \subseteq G(I)$.*

Proof. Let $f \in G(I_z \cap I_y) \cap I$ and $g \in G(I)$ such that $g \mid f$. We can write $g = \text{lcm}(g_z, g_y) \in I_z \cap I_y$, and therefore, since f is a minimal generator of $I_z \cap I_y$ it follows that $f = g$. $\square$

Next, we describe how the minimal generators of the intersection $I_z^n \cap I_y^n$ behave for n larger than the pattern-stability numbers of both I_z and I_y , under the assumption that the sets of weakly persistent generators coincide.

Lemma 5.2.19. *Let I be a monomial ideal in $k[x, y, z]$ with $\text{Min}(R/I) = \{(x, y), (x, z)\}$ such that*

$$P := \{g \in G(I) \mid g_z \in P^*(I_z)\} = \{g \in G(I) \mid g_y \in P^*(I_y)\}.$$

Let $n \in \mathbb{N}$ be greater than or equal to the pattern-stability number of I . Then the following two assertions hold:

- (1) *For every $f \in G(I_z^n \cap I_y^n)$, there exists a $g \in P$ such that $fg \in G(I_z^{n+1} \cap I_y^{n+1})$.*
- (2) *For every $u \in G(I_z^{n+2} \cap I_y^{n+2})$, there exist $f \in G(I_z^{n+1} \cap I_y^{n+1})$ and $g \in P$ such that $u = fg$.*

Proof. We apply the results from Section 3.4 to I_z and I_y such that in both cases the x -degrees of the g_i will determine the link points (that is, we choose $\bullet = x$ in Definition 3.4.4). For this, we first set up the notation for Definition 3.4.4. Let $g_1, \dots, g_{k+1} \in G(I)$ be ordered in descending x -degree such that

$$P_z := P^*(I_z) = \{(g_1)_z, \dots, (g_{k+1})_z\} \text{ and } \\ P_y := P^*(I_y) = \{(g_1)_y, \dots, (g_{k+1})_y\}.$$

We set

$$D := \max \left\{ \left(\mu(I_z) - |P_z| \right) \cdot \delta_{P_z} + |P_z| \cdot d_{P_z}, \left(\mu(I_y) - |P_y| \right) \cdot \delta_{P_y} + |P_y| \cdot d_{P_y} \right\}.$$

Note that $\delta_{P_\bullet}$ and $d_{P_\bullet}$ depend on the (x, y) -degrees of the g_i if $\bullet = z$ and on the (x, z) -degrees of the g_i if $\bullet = y$ (cf. Notations 3.2.9 and 3.2.13). Since n is greater than or equal to the pattern-stability number of I , it follows from Remark 5.2.1 and (5.2.1) that $r := n - D - 1 \geq r_\bullet(P_\bullet, D)$.

For $0 \leq i \leq k+1$ let $\ell_i^\bullet := h_i^x$ be as in the Definition 3.4.4 of the (n, x) -stable components of $I_\bullet$ (with respect to $P_\bullet$ and D). While the dependence of x was essential in Definition 3.4.4, we now apply that definition uniformly with respect to x in both cases. However, to distinguish between the settings I_y and I_z , we use the notational convention ℓ_i^y and ℓ_i^z accordingly.

For $1 \leq i \leq k$ and $\bullet \in \{z, y\}$, the monomial $\ell_i^\bullet$ is defined to be the minimal generator of $I_\bullet^n$ with

$$\deg_x \ell_i^\bullet = \min \{ \deg_x f \mid f \in G(I_\bullet^n), \deg_x f \geq d_i \},$$

where $d_i := r \operatorname{dist}_x(g_i, g_{i+1}) + (r+1) \deg_x g_{i+1}$, see Definition 3.4.4.

For (1), let $f := \operatorname{lcm}(f_1, f_2) \in G(I_z^n \cap I_y^n)$ with $f_1 \in G(I_z^n)$ and $f_2 \in G(I_y^n)$.

Without restriction, we assume that $\deg_x f_2 \leq \deg_x f_1$, that is, $f_2 = h_{f_1}$ with the notation of Proposition 5.2.15. We split into two cases:

Case 1. There exists i such that $d_i \leq \deg_x f_2 \leq \deg_x f_1 \leq d_{i-1}$. In this case

$$\deg_x \ell_i^z \leq \deg_x f_1 \leq \deg_x \ell_{i-1}^z \quad \text{and} \quad \deg_x \ell_i^y \leq \deg_x f_2 \leq \deg_x \ell_{i-1}^y,$$

which, according to Corollary 3.4.18, implies that

$$f_1 \cdot (g_i)_z \in G(I_z^{n+1}) \quad \text{and} \quad f_2 \cdot (g_i)_y \in G(I_y^{n+1}).$$

It now follows from Proposition 5.2.15 that

$$f \cdot g_i = \operatorname{lcm}(f_1 \cdot (g_i)_z, f_2 \cdot (g_i)_y) \in G(I_z^{n+1} \cap I_y^{n+1}).$$

Case 2. There exists i such that $\deg_x f_2 < d_i < \deg_x f_1$. Since $\operatorname{dist}_x(f_2, f_1) \leq \operatorname{dist}_x(\ell_i^y, f_1)$ by Proposition 5.2.15 and $f_2 = \ell_i^y$ is excluded by the degree condition, it follows that $\deg_x f_1 < \deg_x \ell_i^y$. With similar reasoning, we deduce that $i = k$ or $d_{i+1} \leq \deg_x f_2$. Hence, $i = k$ or

$$d_{i+1} \leq \deg_x \ell_{i+1}^y \leq \deg_x f_2 < \deg_x \ell_i^y.$$

In addition, according to Remark 3.4.5 it follows that $\deg_x \ell_i^y \leq d_i + \operatorname{dist}_x(g_i, g_{i+1})$, which further implies

$$d_i \leq \deg_x \ell_i^z \leq \deg_x f_1 < \deg_x \ell_i^y \leq d_i + \operatorname{dist}_x(g_i, g_{i+1}) \leq \deg_x \ell_i^z + \operatorname{dist}_x(g_i, g_{i+1}).$$

It follows by Corollary 3.4.18 that $f_1 \cdot (g_{i+1})_z \in G(I_z^{n+1})$ and $f_2 \cdot (g_{i+1})_y \in G(I_y^{n+1})$. This completes the proof of (1).

For (2), let $u := \operatorname{lcm}(u^z, u^y) \in G(I_z^{n+2} \cap I_y^{n+2})$ with $u^z \in G(I_z^{n+2})$ and $u^y \in G(I_y^{n+2})$. Then there exists an $i \in [k+1]$ such that

$$\deg_x g_{i+1}^{n+2} \leq \deg_x u \leq \deg_x g_i^{n+2}.$$

As in the proof of (1), we denote by $(\ell_i^\bullet)_{i=1}^k$ the link points of the (n, x) -stable components of $I_\bullet$.

By Corollary 3.4.19 it holds that $u^\bullet \in q_\bullet \cdot G(I_\bullet^n)$, where

$$q_\bullet = \begin{cases} (g_{i+1})_\bullet^2 & \text{if } \deg_x u^\bullet \leq \deg_x \ell_i^\bullet g_i g_{i+1} \\ (g_i g_{i+1})_\bullet & \text{if } \deg_x \ell_i^\bullet g_i g_{i+1} \leq \deg_x u^\bullet \leq \deg_x \ell_i^\bullet g_i^2 \\ (g_i)_\bullet^2 & \text{if } \deg_x u^\bullet \geq \deg_x \ell_i^\bullet g_i^2. \end{cases}$$

Therefore if $\deg_x \ell_i^\bullet g_i g_{i+1} \leq \deg_x u^\bullet$ holds for both $\bullet = y$ and $\bullet = z$, then $u \in g_i(I_z^{n+1} \cap I_y^{n+1})$. Similarly, if the reverse inequality $\deg_x \ell_i^\bullet g_i g_{i+1} \geq \deg_x u^\bullet$ holds for both $\bullet = y$ and $\bullet = z$, then $u \in g_{i+1}(I_z^{n+1} \cap I_y^{n+1})$. Thus, the only case left to consider is, without loss of generality, if

$$\deg_x u^y < \deg_x \ell_i^y (g_i g_{i+1})_y \quad \text{and} \quad \deg_x u^z > \deg_x \ell_i^z (g_i g_{i+1})_z.$$

If $\deg_x u^z \leq \deg_x \ell_i^z g_i^2$, then as above it follows that $u \in g_{i+1}(I_z^{n+1} \cap I_y^{n+1})$, hence we consider the case that $\deg_x u^z > \deg_x \ell_i^z g_i^2$. With d_i as defined in the proof of (1), note that

$$\deg_x \ell_i^y \leq d_i + \text{dist}_x(g_1, g_{i+1}) \leq \deg_x \ell_i^z + \text{dist}_x(g_1, g_{i+1})$$

and thus, $\deg_x \ell_i^y g_i g_{i+1} \leq \deg_x \ell_i^z g_i^2$. This further implies that

$$\deg_x u^y < \deg_x \ell_i^y g_i g_{i+1} \leq \deg_x \ell_i^z g_i^2 \leq \deg_x u^z$$

and hence $\text{lcm}(u_1, u_2) \notin G(I_z^{n+1} \cap I_y^{n+1})$ by Proposition 5.2.15, a contradiction

□

We are now ready to prove the main result of this section, which provides a bound on the stability index for ideals with two minimal primes of height two.

Theorem 5.2.20. *Let I be a monomial ideal in $k[x, y, z]$ with $\text{Min}(R/I) = \{(x, y), (x, z)\}$. Then the stability index of I is at most the pattern-stability number of I plus one, i.e.,*

$$\text{stab}(I) \leq \mu(I) \left((\text{dist } I)^2 - 1 \right) + 2.$$

Proof. Write $n := \mu(I) \left((\text{dist } I)^2 - 1 \right) + 1$. By Remark 5.2.1, the pattern-stability numbers of I_z and I_y are both at most n .

If $\{g \in G(I) \mid g_z \in P^*(I_z)\} \neq \{g \in G(I) \mid g_y \in P^*(I_y)\}$, then by Proposition 5.2.14 it follows that $\text{stab}(I) \leq \text{dist}_x I \leq n$. It remains to consider the case

$$\{g \in G(I) \mid g_z \in P^*(I_z)\} = \{g \in G(I) \mid g_y \in P^*(I_y)\}.$$

Let $g_1, \dots, g_{k+1} \in G(I)$ be ordered in descending x -degree such that

$$P^*(I_z) = \{(g_1)_z, \dots, (g_{k+1})_z\} \text{ and} \\ P^*(I_y) = \{(g_1)_y, \dots, (g_{k+1})_y\}.$$

Claim 1: If $\mathfrak{m} \notin \text{Ass}(R/I^{n+1})$, then $\mathfrak{m} \notin \text{Ass}(R/I^{n+1+\ell})$ for all $\ell \geq 0$.

We assume that $\mathfrak{m} \notin \text{Ass}(R/I^{n+1})$, and hence $I^{n+1} = I_z^{n+1} \cap I_y^{n+1}$, see Remark 5.2.12. By the inequality (5.2.1), we can apply Lemma 5.2.19(2). Therefore, every minimal generator u of $I_z^{n+2} \cap I_y^{n+2}$ can be written as $u = f \cdot g_i$ for some $f \in G(I_z^{n+1} \cap I_y^{n+1}) = G(I^{n+1})$ and $i \in \{1, \dots, k+1\}$. It follows that $fg_i \in I^{n+2}$. Since the inclusion $I^{n+2} \subseteq I_z^{n+2} \cap I_y^{n+2}$ is always fulfilled, we can conclude that $I^{n+2} = I_z^{n+2} \cap I_y^{n+2}$ and hence $\mathfrak{m} \notin \text{Ass}(R/I^{n+2})$. Inductively, we obtain that $\mathfrak{m} \notin \text{Ass}(R/I^{n+1+\ell})$ for all $\ell \geq 0$.

Claim 2: If $\mathfrak{m} \in \text{Ass}(R/I^n)$, then $\mathfrak{m} \in \text{Ass}(R/I^{n+\ell})$ for all $\ell \geq 0$.

To prove Claim 2, we assume that $\mathfrak{m} \in \text{Ass}(R/I^n)$. Again by Remark 5.2.12, this is the case if and only if $I^n \subsetneq I_z^n \cap I_y^n$. Hence there exists a monomial $f \in G(I_z^n \cap I_y^n)$ such that $f \notin I^n$. Write $f = \text{lcm}(f_1, f_2)$ for $f_1 \in G(I_z^n)$ and $f_2 \in G(I_y^n)$. Then, by Remark 5.2.17, $f_z = f_1$ or $f_y = f_2$. In both cases we proceed analogously. If $f_z = f_1$, we apply the theory of Section 3.4 to I_z , if $f_y = f_2$, we do the same with I_y . In both cases, $xy = x$ and hence the monomials g_i are already ordered correspondingly. Throughout, we choose $P = P^*(I_z)$ and $P = P^*(I_y)$, respectively. Thus, the two cases differ only by a renaming of variables. Without restriction we assume $f_z = f_1$.

For $0 \leq i \leq k+1$ let $h_i := h_i^x$ as in Definition 3.4.4 with the ideal I_z , $\bullet = x$, $P = P^*(I_z)$, $D = D_P$ and $s = n$ (by (5.2.1), n is large enough such the requirements in Definition 3.4.4 are fulfilled). Recall from Remark 3.4.5 that the monomials h_i are ordered in descending x -degree. Thus, there exists an $i \in \{1, \dots, k\}$ such that

$$\deg_x h_i \leq \deg_x f \leq \deg_x h_{i-1}. \quad (5.2.5)$$

By Lemma 5.2.19(1), $fg_j \in G(I_z^{n+1} \cap I_y^{n+1})$ for some $j \in \{1, \dots, k+1\}$, and by Corollary 3.4.18 and the Conditions (5.2.5) on the x -degree of f , it follows that $i = j$. To simplify notation, we write $g := g_i$.

We claim that $fg \notin I^{n+1}$. If this claim holds true, then $fg \in G(I_z^{n+1} \cap I_y^{n+1}) \setminus I^{n+1}$, and thus $\mathfrak{m} \in \text{Ass}(R/I^{n+1})$. We assume otherwise that $fg \in I^{n+1}$, then by Lemma 5.2.18 it follows that $fg \in G(I^{n+1})$, so we can write

$$fg = f_1 \cdots f_{n+1}, \quad (5.2.6)$$

where $f_i \in G(I)$. We will show that there exists an $i \in \{1, \dots, n+1\}$ such that $f_i = g$, which then leads to the contradiction $f \in I^n$. From (5.2.6) it follows that

$$(fg)_z = f_z g_z = (f_1)_z \cdots (f_{n+1})_z.$$

Among these factors, we group the generators in P together and obtain the representation of $f_z g_z$ as a product

$$f_z g_z = \prod_{p \in P} p^{j_p} \cdot \prod_{q \in N} q^{j_q},$$

where $N := G(I_z) \setminus P$ and

$$\sum_{p \in P} j_p + \sum_{q \in N} j_q = n + 1. \quad (5.2.7)$$

By (5.2.1), we can write

$$n = D + r + 1, \quad (5.2.8)$$

where $D := D_P = |N|\delta_P + |P|d_P$ and $r \geq \min\{r_x(P, D), r_y(P, D)\}$, see Notation 3.4.2.

Note that by Proposition 3.2.3, an element in I_z^{n+1} which is divisible by power q^k with $q \in N$ and $k > \delta_P$ cannot be a minimal generator of I_z^{n+1} . Therefore, since $f_z g_z \in G(I_z^{n+1})$, it follows that $j_q \leq \delta_P$ for all $q \in N$. Hence $\sum_{q \in N} j_q \leq |N|\delta_P$. This inequality, in combination with (5.2.7) and (5.2.8) gives

$$r + 2 + |P| \cdot d_P \leq \sum_{p \in P} j_p.$$

Thus, there must exist at least one $p \in P$ such that $j_p > d_P$. We write

$$B := \{p \in P \mid j_p > d_P\}.$$

Then $B \neq \emptyset$, and we claim that neither $\deg_x p < \deg_x g$ nor $\deg_x p > \deg_x g$ can hold for all $p \in B$. We denote

$$\nu := \prod_{q \in N} q^{j_q} \quad \text{and} \quad \sigma := \prod_{p \in P \setminus B} p^{j_p}.$$

With this notation, $(fg)_z = \nu \cdot \sigma \cdot \prod_{p \in B} p^{j_p}$. Let $n_\nu := \sum_{q \in N} j_q$, and $n_\sigma := \sum_{p \in P \setminus B} j_p$. Then $n_\nu + n_\sigma = D - \ell$ for some $\ell \geq 0$. Furthermore,

$$\sum_{p \in B} j_p = D + r + 2 - (D - \ell) = r + 2 + \ell. \quad (5.2.9)$$

Case 1: Assume that for all $p \in B$, the x -degree of p is less than or equal to the x -degree of g_{i+1} . Then the x -degree of fg fulfills

$$\begin{aligned} \deg_x(fg) &\leq \deg_x(\nu\sigma) + \deg_x g_{i+1} \sum_{p \in B} j_p \leq (D - \ell) \deg_x I + \deg_x g_{i+1} (r + 2 + \ell) \\ &\leq D \deg_x I + (r + 2) \deg_x g_{i+1}, \end{aligned}$$

On the other hand, in (5.2.5), we assumed that $\deg_x f \geq \deg_x h_i$. Further, by the choice of r , we have that

$$r \geq D \cdot \frac{\deg_x I}{\deg_x(g, g_{i+1})},$$

and hence $r \operatorname{dist}_x(g, g_{i+1}) \geq D \operatorname{dist}_x I$. By definition, h_i is the minimal generator of I_z^n with x -degree equal to

$$\deg_x h_i = \min\{\deg_x h \mid h \in G(I_z^n), \deg_x h \geq r \operatorname{dist}_x(g, g_{i+1}) + (r+1) \deg_x g_{i+1}\}.$$

Thus

$$\begin{aligned} \deg_x(fg) &\geq \deg_x h_i + \deg_x g \geq r \operatorname{dist}_x(g, g_{i+1}) + (r+1) \deg_x g_{i+1} + \deg_x g \\ &\geq D \operatorname{dist}_x I + (r+1) \deg_x g_{i+1} + \deg_x g > D \operatorname{dist}_x I + (r+2) \deg_x g_{i+1}, \end{aligned}$$

a contradiction.

Case 2: Assume that for all $p \in B$, the x -degree of p is greater than or equal to the x -degree of g_{i-1} . In this case, we use (5.2.9) to bound the x -degree of fg from below as follows:

$$\deg_x(fg) \geq \deg_x(\nu\sigma) + \deg_x g_{i-1} \sum_{p \in B} j_p \geq (r+2+\ell) \deg_x g_{i-1} \geq (r+2) \deg_x g_{i-1}.$$

By definition, h_{i-1} is the minimal generator of I_z^n with x -degree equal to

$$\deg_x h_{i-1} = \min\{\deg_x h \mid h \in G(I_z^n), \deg_x h \geq r \operatorname{dist}_x(g_{i-1}, g) + (r+1) \deg_x g\},$$

and by Remark 3.4.5, $\deg_x h_{i-1} < (r+1)(\deg_x g + \operatorname{dist}_x(g_{i-1}, g))$. The assumption (5.2.5) that $\deg_x f \leq \deg_x h_i$ now implies that

$$\deg_x(fg) < (r+2) \deg_x g + (r+1) \operatorname{dist}_x(g_{i-1}, g) < (r+2) \deg_x g_{i-1},$$

a contradiction.

Since both Case 1 and Case 2 lead to a contradiction, it follows that either $g_z \in B$ or there exist $p_1, p_2 \in B$ such that g_z lies between p_1 and p_2 .

Note that every $p \in P^*(I_z)$ can be written as $p = (g_\ell)_z$ for some $\ell \in \{1, \dots, k+1\}$. If $j_p > 0$, then there exists a $j \in \{1, \dots, n+1\}$ such that $(f_j)_z = (g_\ell)_z$. Since both f_j and g_ℓ are minimal generators of I , it follows that $f_j = g_\ell$.

- (1) If $g_z \in B$, then in particular $j_{g_z} > 0$ and hence there exists an $\ell \in \{1, \dots, n+1\}$ such that $(f_\ell)_z = g_z$ and $f_\ell = g$, yielding

$$f = \frac{f_1 \cdots f_{n+1}}{g} \in I^n.$$

This contradicts the assumption that $f \notin I^n$.

- (2) Hence, there exist $p_1, p_2 \in B$ such that g_z lies between p_1 and p_2 . By Proposition 3.2.3, there exist $m \leq j_p$ and $\alpha \in \mathbb{N}$ such that $g_z^m \mid p_1^\alpha p_2^{m-\alpha}$. Since $(fg)_z$ is a minimal generator, $g_z^m = p_1^\alpha p_2^{m-\alpha}$ must hold. We write $p_1 = (g_j)_z$ and

$p_2 = (g_\ell)_z$. Then g_j^α and $g_\ell^{m-\alpha}$ appear as factors in the product $f_1 \cdots f_{n+1}$. If $\deg_z g^m < \deg_z g_j^\alpha g_\ell^{m-\alpha}$ then the monomial obtained by replacing $g_j^\alpha g_\ell^{m-\alpha}$ with g^m in the product $f_1 \cdots f_{n+1}$ is in I^{n+1} and strictly divides $f_1 \cdots f_{n+1}$. Hence $\deg_z g^m = \deg_z g_j^\alpha g_\ell^{m-\alpha}$ and thus $g^m = g_j^\alpha g_\ell^{m-\alpha}$. We replace $g_j^\alpha g_\ell^{m-\alpha}$ in the product with g^m and end up with the same contradiction as in (1).

We summarize that $fg \in (I_z^{n+1} \cap I_y^{n+1}) \setminus I^{n+1}$ and therefore $\mathfrak{m} \in \text{Ass}(R/I^{n+1})$. This concludes the proof. $\square$

6. BOUNDS ON THE COPERSISTENT INDEX OF GENERAL MONOMIAL IDEALS¹

As outlined in the introduction, the sequence of associated primes of powers of *any* ideal eventually stabilises. A natural question that arises is whether there exists a universal bound for the stability index at which this stabilisation occurs. Hoa [31] provides an upper bound for the stability index of monomial ideals in terms of the number of variables, the number of generators, and the maximal total degree of the generators (see Fact 2.3.24). In the same work, Hoa illustrates through examples that any such bound necessarily depends on both the number of variables and the degrees of the generators.

However, this bound is in general very large. For instance, for the ideal $I = (xy, yz)$ in $k[x, y, z]$, the bound is greater than $8 \cdot 10^7$, even though the actual stability index is 1; see [6, Example 2.17]. For certain classes of monomial ideals, sharper bounds have been found. Herzog, for example, conjectured that for square-free monomial ideals in r variables, the stability index can be bounded above by $r - 1$, cf. [6, Section 2.3]. A lot of research in that area focuses on edge and cover ideals of graphs; see for example [8, 16, 37, 39, 57, 63]. Also other classes of ideals have been studied over the last decades, cf. [23, 28, 34, 64].

Hoa's strategy for deriving the mentioned upper bound involves bounding separately the indices after which the sequence $(\text{Ass}(\mathbf{R}/I^n))_{n \in \mathbb{N}}$ becomes non-decreasing and non-increasing, respectively. Based on this, we define the **persistence index** $B_{\subseteq}^I$ as the smallest integer such that

$$\text{Ass}(\mathbf{R}/I^n) \subseteq \text{Ass}(\mathbf{R}/I^{n+1}) \text{ for all } n \geq B_{\subseteq}^I.$$

If $B_{\subseteq}^I = 1$, then I fulfills the persistence property, see Definition 2.3.21. Analogously, we define the **copersistence index** $B_{\supseteq}^I$ as the smallest integer such that

$$\text{Ass}(\mathbf{R}/I^n) \supseteq \text{Ass}(\mathbf{R}/I^{n+1}) \text{ for all } n \geq B_{\supseteq}^I.$$

With these definitions, the stability index is the maximum of the persistence and the copersistence index, i.e.,

$$\text{stab}(I) = \max\{B_{\supseteq}^I, B_{\subseteq}^I\}.$$

Hoa [31] proved that for a monomial ideal I , we have

$$B_{\supseteq}^I \leq d(rs + s + d)(\sqrt{r})^{r+1}(\sqrt{2d})^{(r+1)(s-1)},$$

¹This chapter is based on joint work with Clemens Heuberger and Roswitha Rissner [30], published in the Journal of Linear Algebra and its Applications.

where r is the number of variables, s is the number of generators of I and d is the maximal total degree of a minimal generator of I (see Subsection 6.2.3). Beyond this bound, however, little is known about $B_{\geq}^I$ in general.

In this chapter, we develop a general framework to derive upper bounds for the copersistence index $B_{\geq}^I$ of a monomial ideal $I \subseteq k[x_1, \dots, x_r]$. This framework is based on characterising membership in monomial ideals through appropriately chosen systems of linear inequalities.

From a methodological perspective, the central results are Theorem 6.2.7 and Proposition 6.2.8, which together offer a flexible approach to bounding $B_{\geq}^I$. Unlike Hoa's original argument, which intertwines the proof with a particular system of inequalities, our approach decouples the method from any specific system. The advantage of this abstraction is that it allows the derivation of multiple bounds by selecting different inequality systems that satisfy prescribed properties.

We briefly recall some relevant results from Chapter 2. In Section 2.1.3, we showed that via localisation, it suffices to determine whether the maximal ideal $\mathfrak{m} = (x_1, \dots, x_r)$ is associated to I . Remark 2.1.43 gives the following characterisation: Let U_n be one of the sets

$$I^n : \mathfrak{m}, \quad \text{sat}(I^n), \quad \text{or} \quad \text{sat}(I^n) \cap I^{n-1}.$$

Then $\mathfrak{m} \in \text{Ass}(R/I^n)$ if and only if $I^n \not\subseteq U_n$, noting that $I^n \subseteq U_n$ always holds. In other words, the maximal ideal is associated to I^n if and only if the component of degree n of the graded module

$$\bigoplus_{i \geq 0} (U_i/I^i) t^i$$

is nonzero, where t denotes the grading variable.

In Section 6.2, we model this module using systems of linear inequalities of the form $Ax \leq b$. Prior to this, Section 6.1 develops necessary theory regarding the sizes of integer solutions to such systems and interprets these results in the context of graded modules. This approach provides insights into the behaviour of U_n/I^n as n increases and ultimately yields a new upper bound for $B_{\geq}^I$, formalised in Theorem 6.2.11. Finally, Section 6.3 compares this newly derived bound with existing bounds in the literature.

6.1 Graded factor modules related to systems of linear inequalities

With the overall goal in view, we begin by considering a more general framework, establishing the connection to the specific class of ideals at a later stage (see Example 6.1.6 and Section 6.2). The methods developed here build upon techniques discussed by Fields in [15, Section 7].

Convention 6.1.1. Throughout this section, $Ax \leq b$ denotes a system of (componentwise)

inequalities, where $A \in \mathbb{Z}^{m \times \nu}$ and $\mathbf{b} \in \mathbb{N}_0^m$.

Remark 6.1.2. Our focus lies on the non-negative integer solutions of such systems, which can always be enforced by appending the constraints $-I_\nu \mathbf{x} \leq \mathbf{0}$, where I_ν denotes the $\nu \times \nu$ identity matrix. For the sake of readability, however, we omit these additional rows and instead consider the intersection of the solution space with $\mathbb{N}_0^\nu$.

Definition 6.1.3. Let $A \in \mathbb{Z}^{m \times \nu}$. For any $\mathbf{b} \in \mathbb{N}_0^m$, we denote the set of all integer solutions of the system by

$$S_{\mathbf{b}} := \{\mathbf{x} \in \mathbb{N}_0^\nu \mid A\mathbf{x} \leq \mathbf{b}\}.$$

Furthermore, we define the following subset of the polynomial ring $k[W_1, \dots, W_\nu]$:

$$\mathfrak{S}_{\mathbf{b}} := \text{span}_k\{W^{\mathbf{x}} \mid \mathbf{x} \in S_{\mathbf{b}}\}.$$

Remark 6.1.4. Note that all the sets introduced in Definition 6.1.3 depend on the matrix A . However, for the sake of readability, we omit this dependence from the notation.

Remark 6.1.5. We observe that the set $S_0 = \{\mathbf{x} \in \mathbb{N}_0^\nu \mid A\mathbf{x} \leq \mathbf{0}\}$ is a submonoid of $\mathbb{N}_0^\nu$, because $\mathbf{0} \in S_0$ and for $\mathbf{x}_1, \mathbf{x}_2 \in S_0$ we have $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 \leq \mathbf{0}$, hence

$$\mathbf{x}_1 + \mathbf{x}_2 \in S_0. \quad (6.1.1)$$

This implies that $\mathfrak{S}_0$ is a ring.

Note that $S_0 \subseteq S_{\mathbf{b}}$ since $\mathbf{b} \geq \mathbf{0}$. Furthermore, if $\mathbf{x} \in S_0$ and $\mathbf{y} \in S_{\mathbf{b}}$, then it follows that $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} \leq \mathbf{0} + \mathbf{b} = \mathbf{b}$. Hence,

$$\mathbf{x} + \mathbf{y} \in S_{\mathbf{b}} \quad (6.1.2)$$

holds which in turn implies that $\mathfrak{S}_{\mathbf{b}}$ is an $\mathfrak{S}_0$ -module.

Example 6.1.6. Let $I = (x^{a_1}, \dots, x^{a_s})$ be a monomial ideal. Then

$$I^n = \left((x^{a_1})^{k_1} \dots (x^{a_s})^{k_s} \mid k_1, \dots, k_s \in \mathbb{N}_0, n = k_1 + \dots + k_s \right).$$

We want to set up a system of linear inequalities that describes when a monomial $x^{\mathbf{h}}$ is an element of I^n . We have $x^{\mathbf{h}} \in I^n$ if and only if there are $k_1, \dots, k_s \in \mathbb{N}_0$ such that $n = k_1 + \dots + k_s$ and $x^{\mathbf{h}}$ is divisible by $(x^{a_1})^{k_1} \dots (x^{a_s})^{k_s}$. It suffices to demand that $n \leq k_1 + \dots + k_s$, since $I^m \subseteq I^n$ for all $m \geq n$. So we are looking for a non-negative

integer solution of

$$\begin{aligned} a_{1,j}k_1 + \cdots + a_{s,j}k_s - h_j &\leq 0, \\ -(k_1 + \cdots + k_s) + n &\leq 0 \end{aligned}$$

for all $j \in [r]$. In other words, $x^h \in I^n$ if and only if there exists $\mathbf{k} \in \mathbb{N}_0^s$ such that

$$\begin{pmatrix} \boxed{\begin{matrix} a_1 & a_2 & \cdots & a_s \end{matrix}} & \boxed{\begin{matrix} -I_r \end{matrix}} & \boxed{\begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix}} \end{pmatrix} \begin{pmatrix} \boxed{k} \\ \boxed{h} \\ \boxed{n} \end{pmatrix} \leq \mathbf{0}.$$

Given a solution to this system of linear inequalities, the key information we need—the exponents $\mathbf{h}$ and the power n —is stored in the last $r+1$ components of the solution. For the first s components, only their existence matters, not their exact values.

Definition 6.1.7. Let $r \in \mathbb{N}_0$ with $r < \nu$ and $\pi_r: \mathbb{N}_0^\nu \rightarrow \mathbb{N}_0^{r+1}$ be the projection of a ν -tuple onto its last $r+1$ entries, i.e., $\pi_r((x_1, \dots, x_\nu)) = (x_{\nu-r}, \dots, x_\nu)$.

Definition 6.1.8. Let $A \in \mathbb{Z}^{m \times \nu}$. For any $\mathbf{b} \in \mathbb{N}_0^m$, we define

$$\mathcal{H}_{\mathbf{b}} := \text{span}_{\mathbf{k}} \{x^{\pi_r(\mathbf{z})} \mid \mathbf{z} \in S_{\mathbf{b}}\} \subseteq \mathbf{k}[x_1, \dots, x_{r+1}].$$

In particular, $\mathcal{H}_0 = \text{span}_{\mathbf{k}} \{x^{\pi_r(\mathbf{z})} \mid \mathbf{z} \in S_0\}$. By setting $\deg x^{\pi_r(\mathbf{z})} = \pi_0(\mathbf{z})$ we impose a grading on $\mathcal{H}_{\mathbf{b}}$. This gives

$$\mathcal{H}_{\mathbf{b}} = \bigoplus_{n \geq 0} H_{\mathbf{b},n},$$

that is,

$$H_{\mathbf{b},n} = \text{span}_{\mathbf{k}} \{x^{\pi_r(\mathbf{z})} \mid \mathbf{z} \in S_{\mathbf{b}}, \pi_0(\mathbf{z}) = n\}.$$

We call $H_{\mathbf{b},n}$ and $H_{0,n}$ the n -th solution spaces corresponding to $Ax \leq \mathbf{b}$.

Remark 6.1.9. For every $n \in \mathbb{N}_0$ the sets $H_{0,n}$ and $H_{\mathbf{b},n}$ are additive subgroups of $\mathbf{k}[x_1, \dots, x_{r+1}]$ and the following properties hold:

- (1) $H_{0,n} \subseteq H_{\mathbf{b},n}$ since $S_0 \subseteq S_{\mathbf{b}}$ (Remark 6.1.5).
- (2) For all $n, m \in \mathbb{N}_0$ we have that

$$H_{0,m}H_{0,n} \subseteq H_{0,m+n}.$$

holds by Equation (6.1.1) in Remark 6.1.5. Therefore, $\mathcal{H}_0$ is a graded subring of the

graded ring $k[x_1, \dots, x_{r+1}]$ (graded in x_{r+1}).

(3) By Equation (6.1.2) in Remark 6.1.5

$$H_{0,n}H_{b,m} \subseteq H_{b,n+m}$$

holds for all $n, m \in \mathbb{N}_0$, hence $\mathcal{H}_b$ is a graded $\mathcal{H}_0$ -module.

Example 6.1.10. In Example 6.1.6 we have $b = 0$ and $H_{0,n} = I^n x_{r+1}^n$ for all $n \in \mathbb{N}_0$. Therefore, $\mathcal{H}_0$ is equal to the Rees algebra of I , cf. 2.3.7, that is,

$$\mathcal{H}_0 = \bigoplus_{n \geq 0} I^n x_{r+1}^n.$$

6.1.1 Estimates on the generators of the solution spaces

We now aim to analyse the sizes of the generators of $\mathfrak{S}_b$ as an $\mathfrak{S}_0$ -module. We use the fundamental fact from linear programming that any polyhedron can be decomposed into a sum of a finitely generated convex hull and a finitely generated cone, denoted by conv and cone , respectively. More precisely, we use known bounds on the entries of the generators of these components. The following fact summarises this result; for a detailed proof, we refer to [55, Theorem 17.1].

Fact 6.1.11 ([55, Proof of Theorem 17.1]). *Let $A \in \mathbb{Z}^{m \times \nu}$, $b \in \mathbb{Z}_{\geq 0}^m$ and $P = \{x \in \mathbb{Q}_{\geq 0}^\nu \mid Ax \leq b\}$. Let Δ denote the maximum absolute value of the subdeterminants of the matrix $(A \mid b)$.*

Then there exist $z_1, \dots, z_\ell \in P$ and $y_1, \dots, y_s \in S_0$ with all components at most Δ in absolute value such that

$$P = \text{conv}\{z_1, \dots, z_\ell\} + \text{cone}\{y_1, \dots, y_s\}$$

holds.

Moreover, every $x \in S_b$ can be written as $x = \tilde{x} + y$ with $y \in S_0$ and $\tilde{x} \in M \cap S_b$, where

$$M = \text{conv}\{z_1, \dots, z_\ell\} + \left\{ \sum_{i=1}^s \alpha_i y_i \mid 0 \leq \alpha_i < 1, \text{ at most } \nu \text{ of the } \alpha_i \text{ are nonzero} \right\}. \quad (6.1.3)$$

For our purposes, (6.1.3) is crucial: the set M is bounded because the maximum norm of each of $z_1, \dots, z_\ell$ as well as $y_1, \dots, y_s$ is bounded by Δ ; therefore the maximum norm of all vectors in M is bounded by $\Delta(\nu + 1)$. Note that while $z_1, \dots, z_\ell$ might be rational vectors, the set $M \cap S_b$ consists of integer vectors by definition. Rewriting $x = \tilde{x} + y$ in terms of $\mathfrak{S}_0$ and $\mathfrak{S}_b$ immediately leads to the following corollary.

Corollary 6.1.12 (cf. [31, Lemma 2.2]). *Let $Ax \leq b$ be a system as in Convention 6.1.1.*

Then the $\mathfrak{S}_0$ -module $\mathfrak{S}_b$ is generated by finitely many monomials all of whose exponents are at most $\sigma := \Delta(A \mid b)(\nu + 1)$, where $\Delta(A \mid b)$ is the maximum absolute value of the subdeterminants of $(A \mid b)$ and ν is the number of columns of A .

We use Corollary 6.1.12 to get a bound for the degree of the generators of $\mathcal{H}_b$ as an $\mathcal{H}_0$ -module.

Proposition 6.1.13. *Let $Ax \leq b$ be as in Convention 6.1.1. Then $\mathcal{H}_b$ is generated as an $\mathcal{H}_0$ -module by homogeneous elements whose degree is less than or equal to*

$$\sigma := \Delta(A \mid b)(\nu + 1),$$

where $\Delta(A \mid b)$ is the maximal absolute value of the subdeterminants of the matrix $(A \mid b)$.

Proof. To simplify notation, we write π instead of π_r within this proof. We restrict the ring epimorphism

$$\begin{aligned} \varphi: \mathbb{k}[W_1, \dots, W_\nu] &\rightarrow \mathbb{k}[x_1, \dots, x_{r+1}] \\ W^z &\mapsto x^{\pi(z)} \end{aligned}$$

to $\mathfrak{S}_b$ resulting in an epimorphism of additive groups

$$\begin{aligned} \varphi': \mathfrak{S}_b &\rightarrow \mathcal{H}_b \\ W^z &\mapsto x^{\pi(z)}. \end{aligned}$$

A further restriction to the ring $\mathfrak{S}_0$ results in the ring epimorphism

$$\varphi'': \mathfrak{S}_0 \rightarrow \mathcal{H}_0.$$

Let L be the kernel of φ'' . Then $\varphi(L\mathfrak{S}_b) = \varphi(L)\varphi(\mathfrak{S}_b) = \{0\}$. Therefore, $L\mathfrak{S}_b$ is a subgroup of the kernel of φ' . In fact, $L\mathfrak{S}_b$ is an $\mathfrak{S}_0$ -submodule of the kernel of φ' and $\mathfrak{S}_b/L\mathfrak{S}_b$ is an $\mathfrak{S}_0/L$ -module.

By Corollary 6.1.12, there exist elements $z_1, \dots, z_\ell$ with $\pi_0(z_i) \leq \sigma$ for $1 \leq i \leq \ell$ such that $\mathfrak{S}_b$ is generated as an $\mathfrak{S}_0$ -module by $W^{z_1}, \dots, W^{z_\ell}$. Hence $\mathfrak{S}_b/L\mathfrak{S}_b$ is generated as an $\mathfrak{S}_0/L$ -module by $W^{z_1} + L\mathfrak{S}_b, \dots, W^{z_\ell} + L\mathfrak{S}_b$. By the isomorphism induced by φ'' we have

$$\mathfrak{S}_0/L \simeq \mathcal{H}_0$$

and furthermore, since φ' is surjective, we get that $x^{\pi(z_1)}, \dots, x^{\pi(z_\ell)}$ are generators of $\mathcal{H}_b$ as an $\mathcal{H}_0$ -module. The isomorphism and the correlation between the generators is visualised in the commutative diagrams in Figure 6.1.

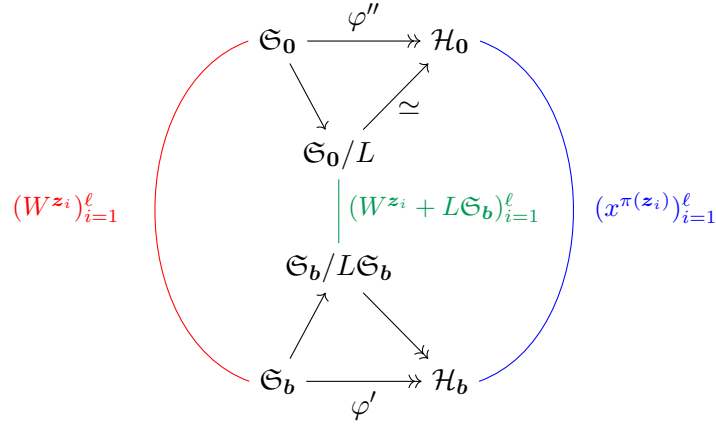


Fig. 6.1: Depiction of the argument in the proof of Proposition 6.1.13 how $\mathfrak{S}_0$ -generators of $\mathfrak{S}_b$ are mapped to $\mathcal{H}_0$ -generators of $\mathcal{H}_b$ via $\mathfrak{S}_0/L$ -generators of $\mathfrak{S}_b/L\mathfrak{S}_b$.

This completes the proof since $\deg(x^{\pi(z_i)}) = \pi_0(z_i) \leq \sigma$ holds for all $i \in \{1, \dots, \ell\}$. $\square$

Remark 6.1.14. The special case of Proposition 6.1.13 applied to $H_{0,n} = I^n$ and $H_{b,n} = \text{sat}(I^n) \cap I^{n-1}$ is already proven by Hoa [31, Lemma 3.3, Lemma 3.4, and Proposition 3.1].

6.1.2 Homogeneous elements of the factor module $\mathcal{H}_b/\mathcal{H}_0$

The ring $\mathcal{H}_0$ is an $\mathcal{H}_0$ -submodule of $\mathcal{H}_b$. A straight-forward verification yields

$$\mathcal{H}_b/\mathcal{H}_0 \simeq \bigoplus_{n \geq 0} (H_{b,n}/H_{0,n}).$$

This is a graded $\mathcal{H}_0$ -module with scalar multiplication

$$h_n \cdot (u_m + H_{0,m}) := h_n u_m + H_{0,m+n}$$

for $h_n \in H_{0,n}$ and $u_m \in H_{b,m}$. Again, the maximal degree of the generators of $\mathcal{H}_b/\mathcal{H}_0$ is bounded by the value σ given in Proposition 6.1.13 because the generators of $\mathcal{H}_b$ as an $\mathcal{H}_0$ -module map to generators of $\mathcal{H}_b/\mathcal{H}_0$ under the projection modulo $\mathcal{H}_0$.

Proposition 6.1.15. *Let $Ax \leq b$ be as in Convention 6.1.1 such that the corresponding solution spaces fulfill $H_{0,m}H_{0,n} = H_{0,m+n}$ for all $n, m \in \mathbb{N}_0$.*

Then the following property holds: If $H_{b,n}/H_{0,n} = 0$ for some $n \geq \sigma = \Delta(A \mid b)(\nu + 1)$, then $H_{b,N}/H_{0,N} = 0$ for all $N \geq n$.

Proof. It suffices to prove that $H_{b,n+1}/H_{0,n+1} = 0$. Recall that the $\mathcal{H}_0$ -module $\mathcal{H}_b$ is generated by elements with degree at most σ according to Proposition 6.1.13. Thus, the homogeneous elements in $H_{b,n+1}/H_{0,n+1}$ are of the form $h_m(u_k + H_{0,k}) = h_m u_k + H_{0,m+k}$, where $k+m = n+1$, $k \leq \sigma$, $h_m \in H_{0,m}$, and $u_k \in H_{b,k}$. Since $h_m \in H_{0,m} = H_{0,1}H_{0,m-1}$

we can write $h_m = ab$, where $a \in H_{0,1}$ and $b \in H_{0,m-1}$. Then

$$h_m u_k + H_{0,m+k} = a \cdot b u_k + H_{0,n+1} = 0,$$

since $b u_k \in H_{b,n} = H_{0,n}$ and $a H_{0,n} \subseteq H_{0,n+1}$. □

6.2 Upper bounds for the copersistence index $B_{\sup}^I$

We now apply the results from the previous section to derive upper bounds for the copersistence index $B_{\sup}^I$. Before doing so, we establish several properties of the parameters d , s and r , on which the resulting bound will depend.

6.2.1 Notes on the bound-parameters d , s , and r

Definition 6.2.1. Let d denote the maximum total degree of the minimal generators of the monomial ideal I and let $\gcd(I) = x^t$ be the greatest common divisor of I . We define the **reduced maximal degree** of I as

$$d_{\text{red}} := d - \sum_{i=1}^r t_i.$$

Remark 6.2.2. We recall Corollary 2.1.49 which states that if a monomial x^t divides all minimal generators of a monomial ideal I , then

$$\text{Ass}(R/I^n) \setminus \{(x_1), \dots, (x_r)\} = \text{Ass}(R/(I : x^t)^n) \setminus \{(x_1), \dots, (x_r)\}$$

holds for all $n \in \mathbb{N}$. In particular, this implies that $\text{stab}(I) = \text{stab}(I : x^t)$. Since the bounds we develop in this section depend on the maximal total degree d of I , and the inequality $d_{\text{red}} \leq d$ always holds, these bounds can be improved by replacing d with d_{red} .

Fact 6.2.3 ([45, Lemma 2.1]). *If the number of generators of a monomial ideal I is smaller than the number of variables, i.e., $s < r$, then $\mathfrak{m} \notin \text{Ass}(R/I^n)$ for all $n \in \mathbb{N}$.*

Remark 6.2.4. The stability index of a monomial ideal in a polynomial ring with two variables ($r = 2$) is equal to 1. This follows from [45, Theorem 2.7], which implies that the maximal ideal $\mathfrak{m}$ is either associated to no powers or to all powers of said ideal, depending on whether it is a principal ideal or not.

6.2.2 Copersistence of prime ideals $\mathfrak{p}(M)$

Recall that for a subset $M \subseteq [r]$, we write $\mathfrak{p}(M) = (x_i \mid i \in M)$.

Definition 6.2.5. Let $I \subseteq k[x_1, \dots, x_r]$ be a monomial ideal and $M \subseteq [r]$. Then we denote by $B_{\sup}^I(M) \in \mathbb{N}$ the smallest number such that the following statement holds: If

for some $N \geq B_{\subseteq}^I(M)$ the prime ideal $\mathfrak{p}(M)$ is not associated to I^N , then it follows that for all $n \geq N$

$$\mathfrak{p}(M) \notin \text{Ass}(R/I^n).$$

Remark 6.2.6. It follows from the definition that $B_{\subseteq}^I = \max \{B_{\subseteq}^I(M) \mid M \subseteq [r]\}$. We argue that in fact

$$B_{\subseteq}^I = \max \{B_{\subseteq}^I(M) \mid M \subseteq [r] \text{ and } |M| \leq s\}$$

where s is the number of generators of I .

Indeed, if $s < |M| \leq r$ then $\mathfrak{m}_{R_M} \notin \text{Ass}(R_M/I_M^n)$ for all n by Fact 6.2.3 since R_M is a polynomial ring in $|M|$ variables (Remark 2.1.28) and the ideal I_M has at most s generators. By Remark 2.1.30 it follows that $\mathfrak{p}(M) \notin \text{Ass}(R/I^n)$ for all n , i.e., $B_{\subseteq}^I(M) = 1$. This proves the claim.

Theorem 6.2.7. Let I be a monomial ideal in $\mathbb{k}[x_1, \dots, x_r]$ and $M \subseteq [r]$. Further, let $A \in \mathbb{Z}^{m \times \nu}$ and $\mathbf{b} \in \mathbb{Z}_{\geq 0}^m$ such that for every $n \in \mathbb{N}$ the associated n -th solution spaces $H_{0,n}$ and $H_{\mathbf{b},n}$ (Definition 6.1.8) fulfill

- (1) $\mathfrak{p}(M) \in \text{Ass}(R/I^n)$ if and only if $H_{\mathbf{b},n}/H_{0,n} \neq 0$, and
- (2) for all $n_1, n_2 \in \mathbb{N}$ we have $H_{0,n_1}H_{0,n_2} = H_{0,n_1+n_2}$.

Then $B_{\subseteq}^I(M) \leq \Delta(A \mid \mathbf{b})(\nu + 1)$.

Proof. This theorem is a special case of Proposition 6.1.15, where the system matrix A and the right-hand side $\mathbf{b}$ are chosen such that the conditions (1) and (2) are satisfied. $\square$

The bound obtained in Theorem 6.2.7 depends on the specific choice of the system $A\mathbf{x} \leq \mathbf{b}$. As a next step, we narrow our focus to the maximal ideal $\mathfrak{m}$, rather than considering all prime ideals $\mathfrak{p}(M)$, having Remark 2.1.30 in mind. For certain systems, it is then possible to further estimate $B_{\subseteq}^I([r])$ by a new bound $\sigma(d, s, r)$ which eliminates the explicit dependence on the system matrix. Instead, this bound depends only on the number of variables r , the number s of generators of I and their maximal total degree d . As the following lemma shows, whenever such a function σ exists and is non-decreasing in each variable, it provides an upper bound for the copersistence index.

Proposition 6.2.8. Let $\sigma: \mathbb{N}^3 \rightarrow \mathbb{N}$ be a map that is non-decreasing in all three variables such that for all $d, s, r \in \mathbb{N}$ the inequality $B_{\subseteq}^I([r]) \leq \sigma(d, s, r)$ holds, whenever I is a monomial ideal in r variables, s generators, and whose minimal generators have total degree at most d . Then $\sigma(d, s, r)$ is an upper bound for the copersistence index $B_{\subseteq}^I$ of every such ideal I .

Proof. Let $M \subseteq [r]$ and I_M the ideal generated by I in the localization R_M of R at $\mathfrak{p}(M)$.

Further, let $\tilde{s}$ the number of minimal generators of I_M and $\tilde{d}$ be their maximal total degree. Then $\tilde{d} \leq d$ and $\tilde{s} \leq s$. Moreover, I_M is an ideal in R_M which is a polynomial ring in $|M| \leq r$ variables. We can conclude that

$$B_{\supseteq}^I(M) = B_{\supseteq}^{I_M}(M) \leq \sigma(\tilde{d}, \tilde{s}, |M|) \leq \sigma(d, s, r),$$

where the leftmost equality is due to Remark 2.1.30 and the middle and rightmost inequality follow from the hypotheses of the proposition. Since $B_{\supseteq}^I = \max\{B_{\supseteq}^I(M) \mid M \subseteq [r]\}$, this finishes the proof. $\square$

In order to find a suitable function σ to bound $B_{\supseteq}^I([r])$, we set up suitable systems $Ax \leq b$ for Theorem 6.2.7 to be applicable with $M = [r]$. In Remark 2.1.43 we gave three statements that characterize $\mathfrak{m} \in \text{Ass}(R/I^n)$. They are of the form

$$\mathfrak{m} \in \text{Ass}(R/I^n) \iff U_n/I^n \neq 0$$

where $U_n \in \{\text{sat}(I^n) \cap I^{n-1}, I^n : \mathfrak{m}, \text{sat}(I^n)\}$. The resulting bounds are discussed below in Subsections 6.2.3, 6.2.4, and 6.2.5. We point out that even for a fixed choice of U_n there are in general multiple options to set up a suitable system $Ax \leq b$. We restrict our investigation to specific choices.

Notation 6.2.9. In contrast to Section 6.1, we write $x = (x_1, \dots, x_r)$ and t instead of x_{r+1} to distinguish notationally between the variables of the ambient ring $k[x_1, \dots, x_r]$ of the ideal I and the variable t we use for the grading of $\mathcal{H}_b$.

6.2.3 Approach 1: $\mathfrak{m} \in \text{Ass}(R/I^n)$ if and only if $(\text{sat}(I^n) \cap I^{n-1}) \neq I^n$

This is the approach followed by Hoa [31], yielding the upper bound for the copersistence index $B_{\supseteq}^I$

$$\sigma_1(d, s, r) := d(rs + s + d)(\sqrt{r})^{r+1}(\sqrt{2}d)^{(r+1)(s-1)},$$

where r is the number of variables, s is the number of generators of I and d is their maximal total degree. For more details, we refer to the original proofs. Proposition 2.1.48 and Remark 6.2.6 further imply

$$B_{\supseteq}^I \leq \sigma_1(d_{\text{red}}, s, \min\{r, s\}).$$

6.2.4 Approach 2: $\mathfrak{m} \in \text{Ass}(R/I^n)$ if and only if $(I^n : \mathfrak{m}) \neq I^n$

We set up a system of linear constraints $Ax \leq b$ such that $H_{b,n} = (I^n : \mathfrak{m})t^n$ and later show that $H_{0,n} = I^n t^n$ (in the notation of Section 6.1 with Notation 6.2.9). A first idea how to set up such a system was introduced in Example 6.1.6.

If $I = (x^{a_1}, \dots, x^{a_s})$, then $I^n = (x^{k_1 a_1 + \dots + k_s a_s} \mid k_i \in \mathbb{N}_0, k_1 + \dots + k_s = n)$. A monomial x^h is an element of $I^n : \mathfrak{m}$ if and only if $x^h x_i \in I^n$ for all $i \in [r]$, i.e., there

exists a generator of I^n that divides $x^h x_i$. That is, for every $i \in [r]$ there exist $k_{i,1}, \dots, k_{i,s} \in \mathbb{N}_0$ such that $k_{i,1} + \dots + k_{i,s} = n$ and

$$x^{k_{i,1}\mathbf{a}_1 + \dots + k_{i,s}\mathbf{a}_s} \mid x^{\mathbf{h} + \mathbf{e}_i},$$

where $\mathbf{e}_i \in \mathbb{Z}^r$ is the i -th unit vector. This is equivalent to the componentwise inequality

$$k_{i,1}\mathbf{a}_1 + \dots + k_{i,s}\mathbf{a}_s \leq \mathbf{h} + \mathbf{e}_i.$$

It suffices to demand that $k_{i,1} + \dots + k_{i,s} \geq n$, since $I^m \subseteq I^n$ holds for all $m \geq n$.

In conclusion: A monomial $x^{\mathbf{h}}$ is an element of $I^n : \mathfrak{m}$ if and only if for every $i \in [r]$ there exist $k_{i,1}, \dots, k_{i,s} \in \mathbb{N}_0$ such that

$$\begin{aligned} k_{i,1}\mathbf{a}_1 + \dots + k_{i,s}\mathbf{a}_s &\leq \mathbf{h} + \mathbf{e}_i, \\ n - (k_{i,1} + \dots + k_{i,s}) &\leq 0. \end{aligned}$$

So for every $i \in [r]$ we get a block of inequalities:

$$\left(\begin{array}{c|c|c} \boxed{\begin{matrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_s \\ -1 & -1 & \dots & -1 \end{matrix}} & \boxed{\begin{matrix} -I_r \\ 0 & \dots & 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix}} \end{array} \right) \begin{pmatrix} \boxed{k_{i,1}} \\ \vdots \\ \boxed{k_{i,s}} \\ \boxed{h_1} \\ \vdots \\ \boxed{h_r} \\ \boxed{n} \end{pmatrix} \leq \begin{pmatrix} \mathbf{e}_i \\ 0 \end{pmatrix}$$

We now combine these blocks to obtain a matrix representation of the system of inequalities.

Notation 6.2.10. Let $m = (r+1)r$ and $\nu = rs + r + 1$. We define the matrix $A_{I^n:\mathfrak{m}} \in \mathbb{Z}^{m+\nu \times \nu}$ and the vector $\mathbf{b}_{I^n:\mathfrak{m}} \in \mathbb{Z}^{m+\nu}$ in the following way:

$$A_{I^n:\mathfrak{m}} := \left(\begin{array}{c|c|c|c} \overbrace{\hspace{1cm}}^s & \overbrace{\hspace{1cm}}^s & \overbrace{\hspace{1cm}}^s & \overbrace{\hspace{1cm}}^{r+1} \\ \hline \boxed{\begin{matrix} \mathbf{a}_1 & \cdots & \mathbf{a}_s \\ -1 & \cdots & -1 \end{matrix}} & & & \boxed{\begin{matrix} -I_r & \boxed{} \\ 0 & \cdots & 0 & 1 \end{matrix}} \\ \hline & \boxed{\begin{matrix} \mathbf{a}_1 & \cdots & \mathbf{a}_s \\ -1 & \cdots & -1 \end{matrix}} & & \boxed{\begin{matrix} -I_r & \boxed{} \\ 0 & \cdots & 0 & 1 \end{matrix}} \\ \hline & & \ddots & \vdots \\ \hline & & \boxed{\begin{matrix} \mathbf{a}_1 & \cdots & \mathbf{a}_s \\ -1 & \cdots & -1 \end{matrix}} & \boxed{\begin{matrix} -I_r & \boxed{} \\ 0 & \cdots & 0 & 1 \end{matrix}} \\ \hline \end{array} \right) \begin{matrix} \left. \vphantom{\begin{matrix} \mathbf{a}_1 & \cdots & \mathbf{a}_s \\ -1 & \cdots & -1 \end{matrix}} \right\} r+1 \\ \left. \vphantom{\begin{matrix} \mathbf{a}_1 & \cdots & \mathbf{a}_s \\ -1 & \cdots & -1 \end{matrix}} \right\} r+1 \\ \\ \left. \vphantom{\begin{matrix} \mathbf{a}_1 & \cdots & \mathbf{a}_s \\ -1 & \cdots & -1 \end{matrix}} \right\} r+1 \end{matrix}$$

where I_r denotes the $r \times r$ identity matrix. Further, we set

$$\mathbf{b}_{I^n:\mathfrak{m}} := (\mathbf{e}_1^\top, 0, \dots, \mathbf{e}_r^\top, 0)^\top \in \mathbb{Z}^{(r+1)r}.$$

Theorem 6.2.11. *Let I be a monomial ideal in the ring $\mathbb{k}[x_1, \dots, x_r]$ with s generators, reduced maximal degree d_{red} (Definition 6.2.1), and*

$$\sigma_2(d, s, r) := (\sqrt{d^2 + 1})^{rs} (\sqrt{r})^{r+2} (rs + r + 2).$$

Then $B_{\geq}^I \leq \sigma_2(d_{\text{red}}, s, \min\{r, s\})$ holds.

Proof. Let $A_{I^n:\mathfrak{m}}$ and $\mathbf{b}_{I^n:\mathfrak{m}}$ be as introduced in Notation 6.2.10. To simplify notation, we write $A = A_{I^n:\mathfrak{m}}$ and $\mathbf{b} = \mathbf{b}_{I^n:\mathfrak{m}}$ in this proof. As explained above, $H_{\mathbf{b},n} = (I^n : \mathfrak{m})t^n$ holds.

Considering the homogeneous system $A\mathbf{x} \leq \mathbf{0}$, the conditions $k_{i,1}\mathbf{a}_1 + \cdots + k_{i,s}\mathbf{a}_s \leq \mathbf{h} + \mathbf{e}_i$ change to $k_{i,1}\mathbf{a}_1 + \cdots + k_{i,s}\mathbf{a}_s \leq \mathbf{h}$, that is, $x^{k_{i,1}\mathbf{a}_1 + \cdots + k_{i,s}\mathbf{a}_s} \mid x^{\mathbf{h}}$. Hence the homogeneous system describes the set of monomials in I^n , that is, $H_{\mathbf{0},n} = I^n t^n$.

This system satisfies the hypotheses of Theorem 6.2.7. Note that $H_{\mathbf{0},n}H_{\mathbf{0},m} = H_{\mathbf{0},n+m}$ holds trivially for all non-negative integers n and m . Therefore,

$$B_{\geq}^I([r]) \leq \Delta(A \mid \mathbf{b})(rs + r + 2)$$

holds.

We use Hadamard's inequality to give an upper bound for $\Delta(A \mid \mathbf{b})$. The norms of the first rs columns of A are at most

$$\max_{i \in [s]} \sqrt{a_{i,1}^2 + \cdots + a_{i,r}^2 + 1} \leq \sqrt{d^2 + 1}.$$

The remaining $r + 1$ columns of A and $\mathbf{b}$ have norm $\sqrt{r}$. Therefore,

$$B_{\subseteq}^I([r]) \leq \Delta(A \mid \mathbf{b})(rs + r + 2) \leq (\sqrt{d^2 + 1})^{rs} (\sqrt{r})^{r+2} (rs + r + 2) = \sigma_2(d, s, r).$$

A straight forward verification shows that σ_2 is non-decreasing in all three parameters. Hence we can apply Proposition 6.2.8 and obtain $B_{\subseteq}^I \leq \sigma_2(d, s, r)$. The assertion follows from Proposition 2.1.48 and Remark 6.2.6. $\square$

6.2.5 Approach 3: $\mathfrak{m} \in \text{Ass}(R/I^n)$ if and only if $\text{sat}(I^n) \neq I^n$

Not too surprisingly, this approach turns out to be very similar to the one we presented in Section 6.2.4. Indeed, the augmented system matrices are almost identical. The resulting bound for $B_{\subseteq}^I$ is greater than $\sigma_2(d_{\text{red}}, s, \min\{r, s\})$ of the previous subsection. However, we briefly describe this approach here to demonstrate that there are several options to construct a system of linear inequalities that is suitable for Theorem 6.2.7.

By definition, $\text{sat}(I^n) = \bigcup_{k \in \mathbb{N}_0} I^n : \mathfrak{m}^k$. As an increasing sequence of ideals in the Noetherian ring R , the sequence $I^n : \mathfrak{m}^0 \subseteq I^n : \mathfrak{m}^1 \subseteq I^n : \mathfrak{m}^2 \subseteq \dots$ becomes stationary at some power $N \in \mathbb{N}$ of $\mathfrak{m}$. Hence, $\text{sat}(I^n) = \bigcup_{k=0}^N I^n : \mathfrak{m}^k$. We will see in Remark 6.2.13 below that the precise value of N is not relevant in what follows. By Remark 2.1.40 we have $\text{sat}(I^n) = \bigcap_{i=1}^r (I^n : x_i^\infty)$ which implies

$$\text{sat}(I^n) = \bigcup_{(k_1, \dots, k_r) \in \mathbb{N}_0^r} I^n : (x_1^{k_1}, \dots, x_r^{k_r}) \supseteq \bigcup_{(k_1, \dots, k_r) \in \mathbb{N}_0^r, k_i \leq N} I^n : (x_1^{k_1}, \dots, x_r^{k_r}).$$

The reverse inclusion also holds, since if $w \in \text{sat}(I^n)$, then there exists a $k \leq N$ such that $w \in I^n : \mathfrak{m}^k$. As $(x_1^k, x_2^k, \dots, x_r^k) \subseteq \mathfrak{m}^k$, this implies that $w \in I^n : (x_1^k, x_2^k, \dots, x_r^k)$. We conclude

$$\begin{aligned} x^{\mathbf{h}} \in \text{sat}(I^n) &\iff \text{there exist } k_1, \dots, k_r \leq N \text{ such that for all } i \in [r], \text{ we have } x^{\mathbf{h}} x_i^{k_i} \in I^n \\ &\iff x^{\mathbf{h}} x_i^N \in I^n \text{ for all } i \in [r]. \end{aligned}$$

This is equivalent to the existence of $k_{i,1}, \dots, k_{i,s} \geq 0$ for all $i \in [r]$ with

$$\begin{aligned} -(k_{i,1} + \dots + k_{i,s}) + n &\leq 0, \\ k_{i,1} \mathbf{a}_1 + \dots + k_{i,s} \mathbf{a}_s &\leq \mathbf{h} + N \mathbf{e}_i. \end{aligned}$$

Notation 6.2.12. Let $A_{I^n:\mathfrak{m}}$ and $\mathbf{b}_{I^n:\mathfrak{m}}$ be as in Notation 6.2.10. Then we set

$$A_{\text{sat}(I^n)} := A_{I^n:\mathfrak{m}} \text{ and } \mathbf{b}_{\text{sat}(I^n)} := N \cdot \mathbf{b}_{I^n:\mathfrak{m}}.$$

Remark 6.2.13. By construction, $H_{\mathbf{b},n} = \text{sat}(I^n)t^n$ holds for the system $A_{\text{sat}(I^n)} \mathbf{x} \leq \mathbf{b}_{\text{sat}(I^n)}$. The same argument as in Subsection 6.2.4 yields $H_{\mathbf{0},n} = I^n t^n$.

Remark 6.2.14. Since $\Delta(A_{\text{sat}(I^n)} \mid \mathbf{b}_{\text{sat}(I^n)}) = \Delta(A_{I^n:\mathbf{m}} \mid N \cdot \mathbf{b}_{I^n:\mathbf{m}}) \geq \Delta(A_{I^n:\mathbf{m}} \mid \mathbf{b}_{I^n:\mathbf{m}})$ holds, using the system $A_{\text{sat}(I^n)} \mathbf{x} \leq \mathbf{b}_{\text{sat}(I^n)}$ in Theorem 6.2.7 and the same technique as in Theorem 6.2.11 does not improve the upper bound for $B_{\geq}^I$ obtained in Theorem 6.2.11.

Remark 6.2.15. As pointed out earlier, there may be more than one choice to set up a system matrix. Another idea was to use that $\text{sat}(I^n) = (I^n : x_1^\infty) \cap \cdots \cap (I^n : x_r^\infty)$. However, the corresponding system is already homogeneous and hence Theorem 6.2.7 is not applicable.

6.3 Comparison of the different approaches

We already established that our approach in Subsection 6.2.5 does not result in a better bound than $\sigma_2(d_{\text{red}}, s, \min\{r, s\})$. It remains to compare the bounds from Subsections 6.2.3 and 6.2.4.

Proposition 6.3.1. *Let $2 \leq r \leq s$ and $d \geq 2$. Then*

$$\sigma_2(d, s, r) < \frac{q(d)^{rs}}{\sqrt{2r}} \cdot \sigma_2(d, s, r) \leq \sigma_1(d, s, r)$$

holds, where $q(d) := \frac{d\sqrt{2}}{\sqrt{d^2+1}} > 1$.

Proof. Due to the hypothesis, we can estimate

$$\begin{aligned} \frac{\sigma_1(d, s, r)}{\sigma_2(d, s, r)} &= \frac{d(rs + s + d) (\sqrt{r})^{r+1} (\sqrt{2}d)^{(r+1)(s-1)}}{(\sqrt{d^2+1})^{rs} (\sqrt{r})^{r+2} (rs + \underbrace{r}_{\leq s} + \underbrace{2}_{\leq d})} \geq \frac{d^{rs+s-r} \sqrt{2}^{rs+s-r-1}}{(\sqrt{d^2+1})^{rs} \sqrt{r}} \\ &= q(d)^{rs} \cdot \frac{d^{s-r} \sqrt{2}^{s-r}}{\sqrt{2r}} \geq \frac{q(d)^{rs}}{\sqrt{2r}}. \end{aligned}$$

This proves the second inequality in the statement of the proposition. For the first inequality, we show that $\frac{q(d)^{rs}}{\sqrt{2r}} > 1$ holds. Since $q(d)$ is increasing in d and $s \geq r$, it follows that

$$\frac{q(d)^{rs}}{\sqrt{2r}} \geq \frac{q(2)^{r^2}}{\sqrt{2r}} =: \varphi(r).$$

The latter expression $\varphi(r)$ is increasing in r , because

$$\frac{q(2)^{r^2}}{\sqrt{2r}} < \frac{q(2)^{r^2+2r+1}}{\sqrt{2(r+1)}} \iff r+1 < q(2)^{2(2r+1)} \cdot r,$$

where the inequality on the right-hand side holds since $q(2)^{2(2r+1)} \geq q(2)^{10} = (8/5)^5 > 10$. Evaluating $\varphi(2)$ gives $64/50 > 1$. $\square$

BIBLIOGRAPHY

- [1] Reza Abdolmaleki and Shinya Kumashiro. Certain monomial ideals whose numbers of generators of powers descend. *Arch. Math. (Basel)*, 116(6):637–645, 2021. doi:10.1007/s00013-021-01596-y.
- [2] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Series in Mathematics. Westview Press, Boulder, CO, economy edition, 2016. For the 1969 original see [MR0242802].
- [3] Shamila Bayati, Jürgen Herzog, and Giancarlo Rinaldo. On the stable set of associated prime ideals of a monomial ideal, 2012. doi:10.1007/s00013-012-0368-0.
- [4] M. Brodmann. The asymptotic nature of the analytic spread. *Math. Proc. Cambridge Philos. Soc.*, 86(1):35–39, 1979. doi:10.1017/S030500410000061X.
- [5] Markus P. Brodmann. Asymptotic stability of $\text{Ass}(M/I^n M)$. *Proc. Amer. Math. Soc.*, 74(1):16–18, 1979. doi:10.2307/2042097.
- [6] Enrico Carlini, Huy Tài Hà, Brian Harbourne, and Adam Van Tuyl. *Ideals of powers and powers of ideals*, volume 27 of *Lecture Notes of the Unione Matematica Italiana*. Springer, Cham, 2020. Intersecting algebra, geometry, and combinatorics, With a foreword by Alfio Ragusa. doi:10.1007/978-3-030-45247-6.
- [7] Marc Chardin. Powers of ideals and the cohomology of stalks and fibers of morphisms. *Algebra Number Theory*, 7(1):1–18, 2013. doi:10.2140/ant.2013.7.1.
- [8] Janet Chen, Susan Morey, and Anne Sung. The stable set of associated primes of the ideal of a graph. *Rocky Mountain J. Math.*, 32(1):71–89, 2002. doi:10.1216/rmj/1030539608.
- [9] David A. Cox, John Little, and Donal O’Shea. *Ideals, varieties, and algorithms*. Undergraduate Texts in Mathematics. Springer, Cham, fourth edition, 2015. An introduction to computational algebraic geometry and commutative algebra. doi:10.1007/978-3-319-16721-3.
- [10] S. Dale Cutkosky, Jürgen Herzog, and Ngô Việt Trung. Asymptotic behaviour of the Castelnuovo-Mumford regularity. *Compositio Math.*, 118(3):243–261, 1999. doi:10.1023/A:1001559912258.
- [11] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry. doi:10.1007/978-1-4612-5350-1.
- [12] David Eisenbud and Craig Huneke. Cohen-Macaulay Rees algebras and their specialization. *J. Algebra*, 81(1):202–224, 1983. doi:10.1016/0021-8693(83)90216-8.
- [13] Shalom Eliahou, Jürgen Herzog, and Maryam Mohammadi Saem. Monomial ideals with tiny squares. *J. Algebra*, 514:99–112, 2018. doi:10.1016/j.jalgebra.2018.07.037.
- [14] Sara Faridi, Huy Tài Hà, Takayuki Hibi, and Susan Morey. Scarf complexes of graphs and their powers. 2024. URL: <https://arxiv.org/abs/2403.05439>, arXiv:2403.05439.
- [15] J. Bruce Fields. Lengths of Tors determined by killing powers of ideals in a local ring. *J. Algebra*, 247(1):104–133, 2002. doi:10.1006/jabr.2001.9020.
- [16] Christopher A. Francisco, Huy Tài Hà, and Adam Van Tuyl. Colorings of hypergraphs, perfect graphs, and associated primes of powers of monomial ideals. *J. Algebra*, 331:224–242, 2011. doi:10.1016/j.jalgebra.2010.10.025.

- [17] Ralf Fröberg. On Stanley-Reisner rings. In *Topics in algebra, Part 2 (Warsaw, 1988)*, volume 26, Part 2 of *Banach Center Publ.*, pages 57–70. PWN, Warsaw, 1990.
- [18] M. R. Garey, D. S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoret. Comput. Sci.*, 1(3):237–267, 1976. doi:10.1016/0304-3975(76)90059-1.
- [19] Oleksandra Gasanova. Monomial ideals with arbitrarily high tiny powers in any number of variables. *Comm. Algebra*, 48(11):4824–4831, 2020. doi:10.1080/00927872.2020.1772276.
- [20] Huy Tài Hà, Hop Dang Nguyen, Ngo Viet Trung, and Tran Nam Trung. Depth functions of powers of homogeneous ideals. *Proc. Amer. Math. Soc.*, 149(5):1837–1844, 2021. doi:10.1090/proc/15083.
- [21] Huy Tài Hà and Adam Van Tuyl. Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers. *J. Algebraic Combin.*, 27(2):215–245, 2008. doi:10.1007/s10801-007-0079-y.
- [22] M. Herrmann, S. Ikeda, and U. Orbanz. *Equimultiplicity and blowing up*. Springer-Verlag, Berlin, 1988. An algebraic study, With an appendix by B. Moonen. doi:10.1007/978-3-642-61349-4.
- [23] Jürgen Herzog and Ayesha Asloob Qureshi. Persistence and stability properties of powers of ideals. *J. Pure Appl. Algebra*, 219(3):530–542, 2015. doi:10.1016/j.jpaa.2014.05.011.
- [24] Jürgen Herzog and Takayuki Hibi. The depth of powers of an ideal. *J. Algebra*, 291(2):534–550, 2005. doi:10.1016/j.jalgebra.2005.04.007.
- [25] Jürgen Herzog and Takayuki Hibi. *Monomial ideals*, volume 260 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2011. doi:10.1007/978-0-85729-106-6.
- [26] Jürgen Herzog, Takayuki Hibi, and Ngô Việt Trung. Symbolic powers of monomial ideals and vertex cover algebras. *Adv. Math.*, 210(1):304–322, 2007. doi:10.1016/j.aim.2006.06.007.
- [27] Jürgen Herzog, Ayesha Asloob Qureshi, and Maryam Mohammadi Saem. The fiber cone of a monomial ideal in two variables. *J. Symbolic Comput.*, 94:52–69, 2019. doi:10.1016/j.jsc.2018.06.022.
- [28] Jürgen Herzog, Asia Rauf, and Marius Vladiu. The stable set of associated prime ideals of a polymatroidal ideal. *J. Algebraic Combin.*, 37(2):289–312, 2013. doi:10.1007/s10801-012-0367-z.
- [29] Jürgen Herzog and Guangjun Zhu. On the fiber cone of monomial ideals. *Arch. Math. (Basel)*, 113(5):469–481, 2019. doi:10.1007/s00013-019-01347-0.
- [30] Clemens Heuberger, Jutta Rath, and Roswitha Rissner. Stabilization of associated prime ideals of monomial ideals—bounding the copersistence index. *Linear Algebra Appl.*, 707:162–186, 2025. doi:10.1016/j.laa.2024.11.020.
- [31] Lê Tuấn Hoa. Stability of associated primes of monomial ideals. *Vietnam J. Math.*, 34(4):473–487, 2006.
- [32] Craig Huneke and Irena Swanson. *Integral closure of ideals, rings, and modules*, volume 336 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.
- [33] Tomáš Kaiser, Matěj Stehlík, and Riste Škrekovski. Replication in critical graphs and the persistence of monomial ideals. *J. Combin. Theory Ser. A*, 123:239–251, 2014. doi:10.1016/j.jcta.2013.12.005.

- [34] Kazem Khashyarmenesh and Mehrdad Nasernejad. On the stable set of associated prime ideals of monomial ideals and square-free monomial ideals. *Comm. Algebra*, 42(9):3753–3759, 2014. doi:10.1080/00927872.2013.793696.
- [35] Vijay Kodiyalam. Asymptotic behaviour of Castelnuovo-Mumford regularity. *Proc. Amer. Math. Soc.*, 128(2):407–411, 2000. doi:10.1090/S0002-9939-99-05020-0.
- [36] Martin Kreuzer and Lorenzo Robbiano. *Computational commutative algebra 1*. Springer-Verlag, Berlin, 2008. Corrected reprint of the 2000 original.
- [37] Ha Minh Lam and Ngo Viet Trung. Associated primes of powers of edge ideals and ear decompositions of graphs. *Trans. Amer. Math. Soc.*, 372(5):3211–3236, 2019. doi:10.1090/tran/7662.
- [38] E. Lasker. Zur Theorie der moduln und Ideale. *Math. Ann.*, 60(1):20–116, 1905. doi:10.1007/BF01447495.
- [39] José Martínez-Bernal, Susan Morey, and Rafael H. Villarreal. Associated primes of powers of edge ideals. *Collect. Math.*, 63(3):361–374, 2012. doi:10.1007/s13348-011-0045-9.
- [40] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.
- [41] Stephen McAdam and Paul Eakin. The asymptotic Ass. *J. Algebra*, 61(1):71–81, 1979. doi:10.1016/0021-8693(79)90306-5.
- [42] Ezra Miller and Bernd Sturmfels. Monomial ideals and planar graphs. In *Applied algebra, algebraic algorithms and error-correcting codes (Honolulu, HI, 1999)*, volume 1719 of *Lecture Notes in Comput. Sci.*, pages 19–28. Springer, Berlin, 1999. URL: https://doi.org/10.1007/3-540-46796-3_3, doi:10.1007/3-540-46796-3_3.
- [43] Ezra Miller and Bernd Sturmfels. *Combinatorial Commutative Algebra*. Graduate Texts in Mathematics. Springer New York, 2004. URL: <https://books.google.at/books?id=CqEHpxbKgv8C>.
- [44] Susan Morey and Rafael H. Villarreal. Edge ideals: algebraic and combinatorial properties. In *Progress in commutative algebra 1*, pages 85–126. de Gruyter, Berlin, 2012.
- [45] Mehrdad Nasernejad and Saeed Rajaei. Detecting the maximal-associated prime ideal of monomial ideals. *Bol. Soc. Mat. Mex. (3)*, 25(3):543–549, 2019. doi:10.1007/s40590-018-0208-8.
- [46] Emmy Noether. Idealtheorie in Ringbereichen. *Math. Ann.*, 83(1-2):24–66, 1921. doi:10.1007/BF01464225.
- [47] Ahad Rahimi. Squarefree monomial ideals with maximal depth. *Czechoslovak Math. J.*, 70(145)(4):1111–1124, 2020. doi:10.21136/CMJ.2020.0171-19.
- [48] Jutta Rath and Roswitha Rissner. Minimal generating sets of large powers of bivariate monomial ideals, 2025. submitted. URL: <https://arxiv.org/abs/2503.21466>, arXiv: 2503.21466.
- [49] Louis J. Ratliff. On asymptotic prime divisors. *Pacific J. Math.*, 111(2):395–413, 1984. URL: <http://projecteuclid.org/euclid.pjm/1102710578>.
- [50] Louis J. Ratliff and David E. Rush. Two notes on reductions of ideals. *Indiana Univ. Math. J.*, 27(6):929–934, 1978. doi:10.1512/iumj.1978.27.27062.
- [51] Louis J. Ratliff, Jr. On prime divisors of I^n , n large. *Michigan Math. J.*, 23(4):337–352 (1977), 1976. URL: <http://projecteuclid.org/euclid.mmj/1029001769>.

- [52] David Rees. Valuations associated with ideals. *Proc. London Math. Soc.* (3), 6:161–174, 1956. doi:10.1112/plms/s3-6.2.161.
- [53] Les Reid, Leslie G. Roberts, and Marie A. Vitulli. Some results on normal homogeneous ideals. *Comm. Algebra*, 31(9):4485–4506, 2003. doi:10.1081/AGB-120022805.
- [54] Günter Scheja and Uwe Storch. *Lehrbuch der Algebra. Teil 1. Mathematische Leitfäden.* [Mathematical Textbooks]. B. G. Teubner, Stuttgart, second edition, 1994. Unter Einschluß der linearen Algebra. [With inclusion of linear algebra]. doi:10.1007/978-3-322-80137-1.
- [55] Alexander Schrijver. *Theory of linear and integer programming.* Wiley-Interscience Series in Discrete Mathematics. John Wiley & Sons, Ltd., Chichester, 1986. A Wiley-Interscience Publication.
- [56] Rodney Y. Sharp. Convergence of sequences of sets of associated primes. *Proc. Amer. Math. Soc.*, 131(10):3009–3017, 2003. doi:10.1090/S0002-9939-03-07038-2.
- [57] Aron Simis, Wolmer V. Vasconcelos, and Rafael H. Villarreal. On the ideal theory of graphs. *J. Algebra*, 167(2):389–416, 1994. doi:10.1006/jabr.1994.1192.
- [58] Pooja Singla. Minimal monomial reductions and the reduced fiber ring of an extremal ideal. *Illinois J. Math.*, 51(4):1085–1102, 2007. URL: <http://projecteuclid.org/euclid.ijm/1258138534>.
- [59] Richard P. Stanley. The upper bound conjecture and Cohen-Macaulay rings. *Studies in Appl. Math.*, 54(2):135–142, 1975. doi:10.1002/sapm1975542135.
- [60] Richard P. Stanley. *Combinatorics and commutative algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 1996.
- [61] Seth Sullivant. Combinatorial symbolic powers. *J. Algebra*, 319(1):115–142, 2008. doi:10.1016/j.jalgebra.2007.09.024.
- [62] Irena Swanson. Primary decompositions. Unpublished survey.
- [63] Naoki Terai and Ngo Viet Trung. On the associated primes and the depth of the second power of squarefree monomial ideals. *J. Pure Appl. Algebra*, 218(6):1117–1129, 2014. doi:10.1016/j.jpaa.2013.11.008.
- [64] Tran Nam Trung. Stability of associated primes of integral closures of monomial ideals. *J. Combin. Theory Ser. A*, 116(1):44–54, 2009. doi:10.1016/j.jcta.2008.04.006.
- [65] Adam Van Tuyl. A beginner’s guide to edge and cover ideals. In *Monomial ideals, computations and applications*, volume 2083 of *Lecture Notes in Math.*, pages 63–94. Springer, Heidelberg, 2013. URL: https://doi.org/10.1007/978-3-642-38742-5_3, doi:10.1007/978-3-642-38742-5_3.
- [66] Rafael H. Villarreal. Cohen-Macaulay graphs. *Manuscripta Math.*, 66(3):277–293, 1990. doi:10.1007/BF02568497.
- [67] Rafael H. Villarreal. *Monomial algebras.* Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, second edition, 2015.
- [68] David L. Wehlau. Constructive invariant theory. 56:377–383, 1994. doi:10.1007/bf02567695.
- [69] Sarah Jo Weinstein and Irena Swanson. Predicted decay ideals. *Comm. Algebra*, 48(3):1089–1098, 2020. doi:10.1080/00927872.2019.1677687.
- [70] Günter M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. doi:10.1007/978-1-4613-8431-1.