56th Austrian Mathematical Olympiad

National Competition—Preliminary Round—Solutions

3rd May 2025

Problem 1. *Let a, b and c be three pairwise different nonnegative real numbers. Prove that*

$$(a+b+c)\Big(\frac{a}{(b-c)^2}+\frac{b}{(c-a)^2}+\frac{c}{(a-b)^2}\Big)>4.$$

(Karl Czakler)

Solution. Since the inequality is symmetric, we can assume without loss of generality that $a > b > c \geq 0$ holds. If we replace (a, b, c) by $(a-c, b-c, 0)$ in the left-hand expression, we note that the value of the first factor $(a+b+c)$ becomes smaller for $c > 0$.

In the second factor

$$\Big(\frac{a}{(b-c)^2}+\frac{b}{(c-a)^2}+\frac{c}{(a-b)^2}\Big),$$

we note that the denominators remain unchanged, while the numerators each become smaller, while remaining positive. The second factor therefore also becomes smaller. We see that the value attained by the expression on the left-hand side is minimal for $c = 0$.

In this case, the inequality reduces to

$$(a+b)\Big(\frac{a}{b^2}+\frac{b}{a^2}\Big)>4,$$

which follows immediately from the arithmetic-geometric means inequality:

$$(a+b)\Big(\frac{a}{b^2}+\frac{b}{a^2}\Big)=\frac{a^2}{b^2}+\frac{b^2}{a^2}+\frac{b}{a}+\frac{a}{b}>4.$$

Equality is not possible, since the values of a und b are assumed to be different.

(Karl Czakler) □

Problem 2. *Let ABC be an acute triangle with $BC > AC$. Let S be the centroid of ABC and let F be the foot of the altitude from C to AB. The median CS intersects the circumcircle k of ABC a second time in P. It also intersects AB in M. The line SF intersects the circumcircle k in Q such that F lies between S and Q.*

Show that M, P, Q and F lie on a circle.

(Karl Czakler)

Solution. Let X denote the second intersection of CF with k, let Y denote the point such that CY is a diameter of k, and let Z denote the second intersection with k of the parallel to AB through C.

By Thales' theorem, $\angle ZYC = \angle CXZ = 90°$, and by construction, $\angle YCX = 90°$. Thus, $CXYZ$ is a rectangle. Since k, CX and YZ lie symmetric with respect to the bisector of segment AB, said bisector is a symmetry axis of rectangle $CYXZ$. Hence, $\overline{CZ} = 2\overline{FM}$. Furthermore, FM and CZ are parallel. Let S' be the intersection of lines ZF and CM. Then triangles $S'FM$ and $S'ZC$ are similar, and since $\overline{CZ} = 2\overline{FM}$, it follows that $CS' : S'M = 2 : 1$. Therefore, $S' = S$, meaning Z, S, F and Q are collinear. It then follows that

$$\angle MFS = \angle SZC = \angle QZC = \angle QPC = \angle QPS$$

From $\angle MFS = \angle QPS$ it follows that F, Q, M and P lie on a common circle.

(Josef Greilhuber) □

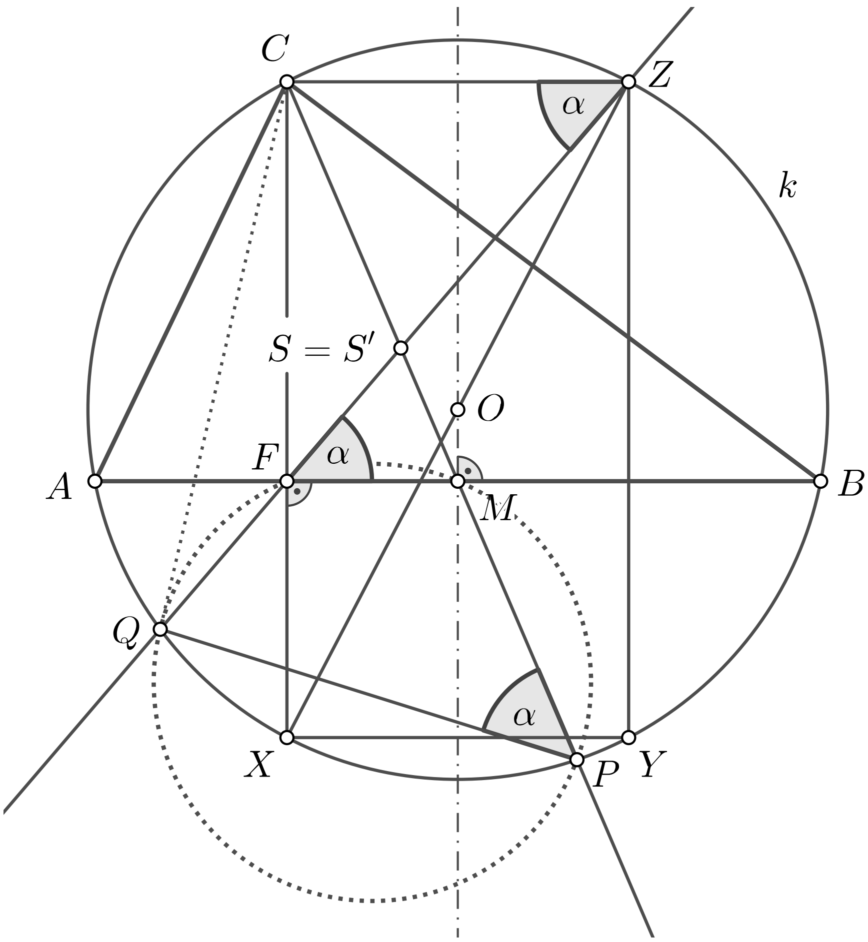

Figure 1: Problem 2

Problem 3. *Consider the following game for a positive integer n: In the beginning, the numbers $1, 2, \ldots, n$ are written on a blackboard. In each step, we select two numbers from the blackboard whose difference is still written on the blackboard. This difference is then erased from the blackboard.*

(For example, if the numbers left on the blackboard are 3, 6, 11 *and* 17, *we can erase* 3 *as* $6-3$ *or* 6 *as* $17-11$ *or* 11 *as* $17-6$.*)*

Using such steps, for which n can we achieve that only one number remains on the blackboard?

(Michael Reitmeir)

Answer. This can be achieved if and only if n is a power of two.

Solution. We first note that n can never be a difference between two numbers, so n must be the last number.

To show that n must be a power of two, we indirectly assume that there is an odd prime number p that divides n, and working backward from the end of the process, we show by induction that p divides all the numbers on the board.

In the end, this is of course true, since p divides the number n. If we now assume, as an induction hypothesis, that at a certain step the only numbers left on the board are those divisible by p, we only need to consider how the last number was erased. If it was the difference between two numbers still on the board, it is of course also divisible by p. If, on the other hand, it was the difference between a number on the board and itself, this other number on the board must be twice the number, since zero is never on the board. However, half of a number divisible by p is also divisible by p.

Since 1 is certainly not divisible by p, we get the desired contradiction.

We now show that powers of two always lead to the goal. To do this, we note that it clearly works for $n = 1$, and for larger powers of two, we first use the pairs $(n, 1)$, $(n, 2)$, …, $(n, n/2 - 1)$, then apply the process inductively to the numbers 1, 2, …, $n/2$ until only $n/2$ remains, and then use the pair $(n, n/2)$.

(Theresia Eisenkölbl) □

Problem 4. *Determine all integers z that can be represented in the form*

$$z = \frac{a^2 - b^2}{b},$$

where a and b are positive integers.

(Walther Janous)

Answer. All integers except for the set $\{\pm 1, \pm 2, \pm 4\}$.

Solution. If z is odd, we write $z^2 = 4k + 1$ (all odd squares are congruent to 1 modulo 4) and set $a = k$, $b = k + \frac{1-z}{2}$. Then we have

$$(2a)^2 + z^2 = 4k^2 + 4k + 1 = (2k + 1)^2 = (2b + z)^2 = 4b^2 + 4bz + z^2,$$

which simplifies to $z = \frac{a^2-b^2}{b}$. So we only have to make sure that a and b are positive integers. First, for $z \neq \pm 1$, we have $4k + 1 = z^2 > 1$, thus $a = k > 0$. For negative z, $b > a > 0$. For positive $z \neq 1$, one obtains $(2k + 1)^2 = 4k^2 + 4k + 1 > 4k + 1 = z^2$, thus $2k + 1 > z$ and $b = \frac{2k+1-z}{2} > 0$. So this procedure works for all odd z except for 1.

In the case $z = 8$, we see that $a = 3$, $b = 1$ is a solution. In the case $z = -8$, the choice $a = 3$, $b = 9$ works. For $z = 0$, we have the solution $a = b = 1$.

All integers $z \notin \{-4, -2, -1, 0, 1, 2, 4\}$ can be written as $z = 2^\ell u$, where u is one of the numbers for which we already have a representation (i.e., u is either equal to 8 or odd). Now we write $u = \frac{a^2-b^2}{b}$ and obtain $z = \frac{(2^\ell a)^2 - (2^\ell b)^2}{2^\ell b}$, which means that z also has a representation.

The cases where $z \in \{\pm 1, \pm 2, \pm 4\}$ lead to the equations

$$(2a)^2 + 1 = (2b \pm 1)^2, \qquad a^2 + 1 = (b \pm 1)^2, \qquad a^2 + 4 = (b \pm 2)^2,$$

which all imply $a = 0$, since the gaps between perfect squares increase. The only perfect squares whose distance is 1 are 0 and 1, and the only perfect squares whose distance is 4 are 0 und 4. Thus there are no solutions in these cases.

(Josef Greilhuber) □