55^{th} Austrian Mathematical Olympiad

National Competition—Preliminary Round—Solutions

27th April 2024

Problem 1. *Let α and β be real numbers with $\beta \neq 0$. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that*

$$f(\alpha f(x) + f(y)) = \beta x + f(y)$$

holds for all real x and y.

(Walther Janous)

Answer. The functional equation only has solutions for $\alpha = \beta$, namely

- for $\alpha = \beta = -1$ the functions $f(x) = x + C$ with $C \in \mathbb{R}$ and
- for $\alpha = \beta \neq 0, -1$ the function $f(x) = x$.

Solution. The function f is injective using the variable x (on the left x only occurs as $f(x)$, on the right x is free with a non-vanishing factor, so substituting $x = a$ and $x = b$ with $f(a) = f(b)$ gives the desired conclusion).

We set $x = 0$ and remove the outer f due to the injectivity and obtain $f(y) = y + C$.

Substituting into the original equation shows that this is equivalent to $\alpha = \beta$ (coefficient of x) and $(1 + \alpha)C = 0$ (constant coefficient).

This gives the solutions $f(x) = x$ for $\alpha = \beta$ and $f(x) = x + C$ for $\alpha = \beta = -1$.

(Theresia Eisenkölbl) □

Problem 2. *Let h be a semicircle with diameter AB. The two circles k_1 and k_2, $k_1 \neq k_2$, touch the segment AB at the points C and D, respectively, and the semicircle h from the inside at the points E and F, respectively. Prove that the four points C, D, E and F lie on a circle.*

(Walther Janous)

Solution. We first consider the case where C and D are both not the center of AB, so that the tangents in C and D are both not parallel to AB.

The tangent in E intersects AB in X, the tangent in F intersects AB in Y and the two tangents intersect each other in Z. Let I now be the intersection point of the angle bisector of $\angle XYZ$ and $\angle ZXY$. Since the tangent segments XC and XE at k_1 are of equal length and C, E lie on the legs of the angle $\angle ZXY$, C and E are equidistant from I.

The same applies to D and F with the circle k_2 and E and F with the semicircle h.

This means that the four points lie on a circle with center I.

In the remaining special case that k_1 passes through the center of AB, we can still define Y as the intersection of the tangent in F with AB, and Z as the intersection of the tangents in E and F. We define I as the intersection of the angle bisectors of $\angle ZYD$ and $\angle EZY$. Therefore, I has the same distance to DY and YZ, and the same distance to EZ and YZ. This means that I also has the same distance to the parallel lines DY and EZ. Thus I lies on the perpendicular bisector of CE and, thus, $IC = IE$.

(Theresia Eisenkölbl) □

Problem 3. *Let $n \geq 3$ be an integer. A* circle dance *is a dance that is performed according to the following rule: On the floor, n points are marked at equal distances along a large circle. At each of these points is a sheet of paper with an arrow pointing either clockwise or counterclockwise. One of the points is labeled „Start". The dancer starts at this point. In each step, he first changes the direction of the arrow at his current position and then moves to the next point in the new direction of the arrow.*

- *a) Show: Each circle dance visits each point infinitely often.*
- *b) How many different circle dances are there? Two circle dances are considered to be the same if they differ only by a finite number of steps at the beginning and then always visit the same points in the same order. (The common sequence of steps may begin at different times in the two dances.)*

(Birgit Vera Schmidt)

Solution. a) By the pigeon-hole principle, there exists at least one point that is visited infinitely often. If there is another point that is visited only finitely many times, then there are also two neighboring points where one point is visited infinitely many times and the other one finitely many times. But this is not possible because the dancer leaves the point that is visited infinitely many times, alternately in the two directions, so he also visits the neighboring points infinitely many times.

b) Claim: If the dancer takes exactly $k < n$ consecutive steps in one direction right before a change of direction, then he takes at least $k + 1$ steps in the other direction after the change of direction.

Proof: After the dancer takes k steps in one direction and changes direction, he first takes one step in the other direction. Because of the previous k steps, he has k arrows in front of him that point toward him. This means that he will certainly take k more steps in the other direction than the first one. □

Therefore, after at most n changes of direction, the dancer will take n consecutive steps in the same direction. With the nth step, he visits the first point of the step sequence, flips the arrow and then has only $n - 1$ arrows in front of him pointing towards him, so he will again make n consecutive steps in one direction.

So we have seen, that every dance eventually has a „turning point". The dancer will dance a whole circle clockwise from the turning point to itself, then a whole circle counter-clockwise from the turning point to itself, and so on.

It is possible to choose the arrow directions at the beginning so that any point can become the turning point. For example, we can have all arrows starting at the start point and continuing counter-clockwise until the desired turning point pointing clockwise and all other arrows pointing counter-clockwise.

Therefore, we have n different dances.

(Birgit Vera Schmidt) □

Problem 4. *A positive integer is called* powerful *if all exponents in its prime factorization are ≥ 2.*

Prove that there are infinitely many pairs of powerful consecutive positive integers.

(Walther Janous)

Solution. The numbers $8 = 2^3$ and $9 = 3^2$ form a pair of consecutive powerful numbers.

We now show that for each pair $(k, k+1)$ of powerful positive integers we can find a new pair, namely the pair $(4k(k + 1), (2k + 1)^2)$. Obviously, $4k(k + 1) + 1 = (2k + 1)^2$. Since k and $k + 1$ are powerful, the product $4k(k + 1) = 2^2k(k + 1)$ is also powerful. And a square is certainly powerful, so in particular $(2k + 1)^2$. Finally, $4k(k + 1) > k$ for positive integers k. This means that there are infinitely many pairs of powerful consecutive positive integers.

(Walther Janous) □