# $52^{\text{nd}}$ Austrian Mathematical Olympiad

National Competition—Preliminary Round—Solutions

1st May 2021

**Problem 1.** *Let $a$, $b$, $c$ be positive real numbers with $a+b+c=1$.*

*Prove that*

$$\frac{a}{2a+1}+\frac{b}{3b+1}+\frac{c}{6c+1}\le\frac{1}{2}.$$

*When does equality hold?*

*(Karl Czakler)*

*Solution.* We will use the following inequalities:

$$\frac{a}{2a+1}\le\frac{2a+1}{8},\qquad\frac{b}{3b+1}\le\frac{3b+1}{12}\qquad\text{and}\qquad\frac{c}{6c+1}\le\frac{6c+1}{24}.$$

They are an immediate consequence of the arithmetic-geometric mean inequality with the pairs of values 1 and $2a$, 1 and $3b$, and 1 and $6c$, so that equality holds for $a=1/2$, $b=1/3$ and $c=1/6$.

From these inequalities and the condition $a+b+c=1$, we obtain

$$\frac{a}{2a+1}+\frac{b}{3b+1}+\frac{c}{6c+1}\le\frac{2a+1}{8}+\frac{3b+1}{12}+\frac{6c+1}{24}=\frac{1}{2}.$$

Therefore, the given inequality is true and equality holds exactly for $a=1/2$, $b=1/3$ and $c=1/6$.

*(Karl Czakler)* □

**Problem 2.** *Let $ABC$ denote a triangle. The point $X$ lies on the extension of $AC$ beyond $A$, such that $AX=AB$. Similarly, the point $Y$ lies on the extension of $BC$ beyond $B$ such that $BY=AB$.*

*Prove that the circumcircles of $ACY$ and $BCX$ intersect a second time in a point different from $C$ that lies on the bisector of the angle $\angle BCA$.*

*(Theresia Eisenkölbl)*

*Solution.* As usual, we denote the angles of the triangle at $A$, $B$ and $C$ with $\alpha$, $\beta$ and $\gamma$.

It is sufficient to show that the center $I_c$ of the excircle touching the line $AB$ lies on the two circles. To do this, we look at the respective inscribed angles.

Since the triangle $AYB$ is isosceles, the following holds:

$$\angle CYA=\angle BYA=90^\circ-\angle YBA/2=90^\circ-90^\circ+\beta/2=\beta/2.$$

But it is also true that

$$\angle CI_cA=180^\circ-(180^\circ-\alpha)/2-\alpha-\gamma/2=90^\circ-\alpha/2-\gamma/2=\beta/2.$$

So $I_c$ lies on the circumcircle of $ACY$ by the inverse of the inscribed angle theorem. In the same way, one also obtains that $I_c$ lies on the circumcircle of $BCX$. So $I_c$ is the second point of intersection, which therefore lies on the angle bisector through $C$ as required.

*(Theresia Eisenkölbl)* □

**Problem 3.** *Let $n \geq 3$ be an integer.*

*On a circle, there are $n$ points. Each of them is labelled with a real number at most 1 such that each number is the absolute value of the difference of the two numbers immediately preceding it in clockwise order.*

*Determine the maximal possible value of the sum of all numbers as a function of $n$.*

*(Walther Janous)*

*Solution.* All the numbers are absolute values, so they are positive or zero. Either all of them are zero, then their sum is also zero, or there is a maximal positive number. If we scale all numbers such that this maximum is 1, the sum can only get larger, therefore, we may assume that the maximum is 1 in this case.

We observe that the difference of two such neighboring numbers smaller than 1 is also smaller than 1. If we iterate around the circle, we see that all numbers have to be smaller than 1 which is impossible if the maximum is 1.

Therefore, in the case of maximum 1, at least one of each pair of neighbors must equal 1. We can now distinguish two cases for the two numbers after the maximum. Either the number immediately after the maximum is also 1 which means that the list of differences continues as $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$ or the next numbers are $1, x, 1$. However this means that $|1 - x| = 1$ so that $x = 0$ and we get the list $1, 0, 1, 1, 0, 1, 1, 0, 1, \ldots$.

In both these subcases, $n$ has to be divisible by 3 to wrap correctly around the circle with these patterns. Then, we get a sum of $2n/3$.

In conclusion, we get maximal sum 0 if $n$ is not divisible by 3 and $2n/3$ if $n$ is divisible by 3.

*(Theresia Eisenkölbl)* □

**Problem 4.** *On a blackboard, there are 17 integers not divisible by 17. Alice and Bob play a game. Alice starts and they alternately play the following moves:*

- *Alice chooses a number $a$ on the blackboard and replaces it with $a^2$.*
- *Bob chooses a number $b$ on the blackboard and replaces it with $b^3$.*

*Alice wins if the sum of the numbers on the blackboard is a multiple of 17 after a finite number of steps.*

*Prove that Alice has a winning strategy.*

*(Daniel Holmes)*

*Solution.* Since both the problem statement and the winning condition are given in terms of divisibility by 17, it is sufficient to consider the numbers modulo 17. In the beginning, all the remainders are different from zero and Alice wins if the sum modulo 17 becomes zero.

The moves $a \mapsto a^2$ and $b \mapsto b^3$ turn remainders into powers of the original nonzero values. Therefore, Fermat's little theorem can be applied. For $a \not\equiv 0 \pmod{17}$ and the prime number 17, one has

$$a^{16} \equiv 1 \pmod{17}.$$

So if Alice squares the same number $a$ four times in a row, then the remainder 1 modulo 17 is always obtained. Bob cannot do anything about it, because if Bob raises this number to the third power $k$ times, we get a result of

$$a^{2^4 \cdot 3^k} = 16^{3^k} \equiv 1^{3^k} = 1 \pmod{17}.$$

The timing of Bob's moves does not matter, as the order of the factors in the exponent does not change anything.

Therefore, Alice can make all the remainders equal to 1 by squaring each number four times. Then of course the sum is $17 \cdot 1 \equiv 0 \pmod{17}$ and Alice has won.

*(Theresia Eisenkölbl)* □