50^{th} Austrian Mathematical Olympiad

National Competition—Preliminary Round—Solutions

4th May 2019

Problem 1. *We consider the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ which are defined by $a_0 = b_0 = 2$ and $a_1 = b_1 = 14$ and by*

$$a_n = 14a_{n-1} + a_{n-2},$$
$$b_n = 6b_{n-1} - b_{n-2}$$

for $n \geq 2$.

Decide whether there are infinitely many integers which occur in both sequences.

(Gerhard Woeginger)

Answer. Yes.

Solution. Sequence (a_n) starts with values 2, 14, 198, 2786, 39202, 551614. Sequence (b_n) starts with values 2, 14, 82, 478, 2786, 16238, 94642, 551614. We therefore conjecture that $a_{2k+1} = b_{3k+1}$ holds for $k \geq 0$.

Shifting the recurrence yields

$$\begin{aligned} a_{n+2} - 14a_{n+1} - a_n &= 0, \\ a_{n+1} - 14a_n - a_{n-1} &= 0, \\ a_n - 14a_{n-1} - a_{n-2} &= 0 \end{aligned}$$

for $n \geq 2$. Multiplying these recurrences by 1, 14 and -1, respectively, and taking the sum yields $a_{n+2} - 198a_n + a_{n-2} = 0$ and thus

$$a_{n+2} = 198a_n - a_{n-2}$$

for $n \geq 2$.

Shifting the recurrence of (b_n) yields

$$\begin{aligned} b_{n+3} - 6b_{n+2} + b_{n+1} &= 0, \\ b_{n+2} - 6b_{n+1} + b_n &= 0, \\ b_{n+1} - 6b_n + b_{n-1} &= 0, \\ b_n - 6b_{n-1} + b_{n-2} &= 0, \\ b_{n-1} - 6b_{n-2} + b_{n-3} &= 0 \end{aligned}$$

for $n \geq 3$. Multiplying these recurrences by 1, 6, 35, 6 and 1, respectively, and taking the sum yields $b_{n+3} - 198b_n + b_{n-3} = 0$ and thus

$$b_{n+3} = 198b_n - b_{n-3}$$

for $n \geq 3$.

We see that the subsequences (a_{2k+1}) and (b_{3k+1}) have the same initial values $a_1 = b_1 = 14$ and $a_3 = b_4 = 2786$ and fulfil the same recurrence. This implies that $a_{2k+1} = b_{3k+1}$ for all $k \geq 0$.

From the given recurrence, it is obvious that the sequence (a_n) is strictly increasing. Thus we also get infinitely many values which occur in both sequences.

(Gerhard J. Woeginger) □

Problem 2. *Let ABC be a triangle and I its incenter. The circumcircle of ACI intersects the line BC a second time in the point X and the circumcircle of BCI intersects the line AC a second time in the point Y.*

Prove that the segments AY and BX are of equal length.

(Theresia Eisenkölbl)

Solution. We shall show that $AB = BX$ holds. Since $AB = AY$ then follows by the same argument, this completes the proof (see Figure 1).

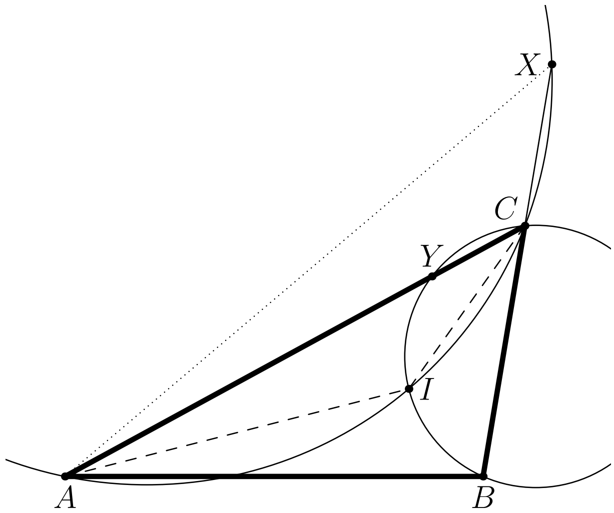

Figure 1: Problem 2

In this solution, we use oriented angles between lines (modulo 180°) with the notation $\angle PQR$. As usual the angles of the triangle ABC are denoted by $\alpha = \angle BAC$, $\beta = \angle CBA$ and $\gamma = \angle ACB$.

The inscribed angle theorem gives

$$\angle AXB = \angle AXC = \angle AIC = -\angle CIA = 180^\circ - \angle CIA = \angle IAC + \angle ACI = \frac{1}{2}(\alpha + \gamma).$$

This immediately implies

$$\angle BAX = -\angle AXB - \angle XBA = -\frac{1}{2}(\alpha + \gamma) - \beta = \frac{1}{2}(\alpha + \gamma).$$

Therefore, the triangle ABX is indeed isosceles, and we are done.

(Theresia Eisenkölbl) □

Problem 3. *Let $n \geq 2$ be an integer.*

Ariane and Bérénice play a game on the set of residue classes modulo n. In the beginning, the residue class 1 is written on a piece of paper. In each move, the player whose turn it is replaces the current residue class x with either $x + 1$ or $2x$. The two players alternate with Ariane starting.

Ariane has won if the residue class 0 is reached during the game. Bérénice has won if she can permanently avoid this outcome.

For each value of n, determine which player has a winning strategy.

(Theresia Eisenkölbl)

Answer. Ariane wins for $n = 2$, 4 and 8, for all other $n \geq 2$ Bérénice wins.

Solution. We observe: If Ariane can win for a certain n, she will also win for all divisors of n, and conversely, if Bérénice can win for a certain n, she will also win for all multiples of n because a residue 0 modulo n is automatically a residue 0 for all divisors of n.

It remains to show that Ariane wins for $n = 8$ and Bérénice wins for $n = 16$ and n odd.

All congruences in this solution are modulo n.

- For $n = 8$, Ariane has to choose 2 in the first step. If Bérénice takes 4, Ariane can choose $8 \equiv 0$ and has won. If Bérénice takes 3, Ariane can choose 6. Now, Bérénice has to decide between 7 and $2 \cdot 6 = 12 \equiv 4$. But for both, Ariane can immediately choose $8 \equiv 0$.

- For $n = 16$, Bérénice chooses $2x$ for all numbers except 4 and 8. This clearly never gives the residue classes 0, 15 or 8, so that Ariane also cannot choose 0.

- For $n = 3$, Ariane has to choose 2 in the first step and then Bérénice chooses 1 again, which means that Bérénice wins.

- For odd $n > 3$, it is not possible to reach 0 with $2x$ from another residue class. So the only possible issue for Bérénice would be the situation that both her options are among n and $n-1$ such that she or Ariane choose 0. But this means that $x + 1$ takes the residues 0 or -1, so $2x$ takes the residues -2 or -4 which are both different from 0 and -1, so this cannot happen and Bérénice can permanently avoid 0 being chosen.

(Theresia Eisenkölbl) □

Problem 4. *Find all pairs (a, b) of real numbers such that*

$$a \cdot \lfloor b \cdot n \rfloor = b \cdot \lfloor a \cdot n \rfloor$$

for all positive integers n.

(Walther Janous)

Answer. The solutions are all pairs (a, b) with $a = 0$ or $b = 0$ or $a = b$ or both a and b integers.

Solution. Let $a_0 = \lfloor a \rfloor$ and a_i be the binary digits of the fractional part of a such that $a = a_0 + \sum_{i=1}^{\infty} \frac{a_i}{2^i}$ with $a_0 \in \mathbb{Z}$ and $a_i \in \{0, 1\}$ for $i \geq 1$. Similarly, let $b = b_0 + \sum_{i=1}^{\infty} \frac{b_i}{2^i}$ with $b_0 \in \mathbb{Z}$ and $b_i \in \{0, 1\}$ for $i \geq 1$. In the case of a non-unique binary expansion, we choose the expansion ending on infinitely many zeros.

Now choose $n = 2^k$ and $n = 2^{k-1}$ in the given equation. We get the equations

$$a\left(2^k b_0 + \sum_{i=1}^{k} b_i 2^{k-i}\right) = b\left(2^k a_0 + \sum_{i=1}^{k} a_i 2^{k-i}\right),$$

$$a\left(2^{k-1} b_0 + \sum_{i=1}^{k-1} b_i 2^{k-i-1}\right) = b\left(2^{k-1} a_0 + \sum_{i=1}^{k-1} a_i 2^{k-i-1}\right).$$

The first equation for $k = 0$ and the difference of the first equation and the doubled second equation for $k \geq 1$ yields

$$ab_k = ba_k \tag{1}$$

for $k \geq 0$.

Now, we consider three cases. If one or both of a and b are zero, then the original equation is clearly satisfied. If both fractional parts are zero, then both numbers are integers and again, the original equation is satisfied. So, finally, we consider the case that a, $b \neq 0$ and that there is a $k \geq 1$ with $a_k = 1$. The equation (1) shows that b_k cannot be zero, so we get $b_k = 1$ and thus from the same equation $a = b$. This clearly satisfies the original equation. (Of course, $b_k = 1$ leads to the same conclusion.) Therefore, the solutions are exactly the pairs listed in the answer.

(Theresia Eisenkölbl) □