49^{th} Austrian Mathematical Olympiad

National Competition (Final Round, part 1)—Solutions

28th April 2018

Problem 1. *Let α be an arbitrary positive real number. Determine for this number α the greatest real number C such that the inequality*

$$\left(1+\frac{\alpha}{x^2}\right)\left(1+\frac{\alpha}{y^2}\right)\left(1+\frac{\alpha}{z^2}\right) \geq C \cdot \left(\frac{x}{z}+\frac{z}{x}+2\right)$$

is valid for all positive real numbers x, y and z satisfying $xy+yz+zx=\alpha$. When does equality occur?

(Walther Janous)

Solution. By replacing α by $xy+yz+zx$ and clearing fractions we get the equivalent inequality

$$(x^2+xy+xz+yz)(y^2+yx+yz+xz)(z^2+zx+zy+xy) \geq Cxy^2z(x^2+z^2+2xz).$$

This inequality is homogeneous of degree 6, thus no further constraint has to be considered. As each of the three factors on the left-hand-side can be factorized we get

$$(x+y)(x+z)(y+x)(y+z)(z+x)(z+y) \geq Cxy^2z(x+z)^2.$$

Upon cancellation of $(x+z)^2$ we arrive at the equivalent inequality

$$(x+y)^2(z+y)^2 \geq Cxy^2z.$$

Estimating each of the two factors on the left with the arithmetic-geometric inequality we obtain the optimal value of $C=16$ which is attained by $x=y=z=\sqrt{\alpha/3}$.

(Clemens Heuberger) □

Problem 2. *Let ABC be a triangle with incenter I. The incircle of the triangle is tangent to the sides BC and AC in points D and E, respectively. Let P denote the common point of lines AI and DE, and let M and N denote the mid-points of sides BC and AB, respectively. Prove that points M, N and P are collinear.*

(Karl Czakler)

Solution. For $AB=AC$, we get $D=M=P$, so the points M, N and P are trivially collinear. We will now only prove the case $AB>AC$ as $AB<AC$ is completely analogous.

Let α, β and γ denote the interior angles of the triangle in A, B and C respectively, as usual; see Figure 1. We note that

$$\angle BDP = \angle CDE = 90° - \frac{\gamma}{2} \qquad \text{and} \qquad \angle BIP = \frac{\alpha+\beta}{2} = 90° - \frac{\gamma}{2}$$

certainly hold, which implies that the quadrilateral $BPDI$ is inscribed. We therefore have $90° = \angle IDB = \angle IPB$, which implies that the triangle APB is right. Since N is the mid-point of its hypotenuse, it must be its circumcenter, which yields $\angle BNP = 2\cdot\angle BAP = \alpha$. We see that PN is parallel to AC, and since MN is also parallel to AC by virtue of the fact that M and N are the mid-points of their respective sides of ABC, we see that M, N and P must be collinear, as claimed.

(Karl Czakler) □

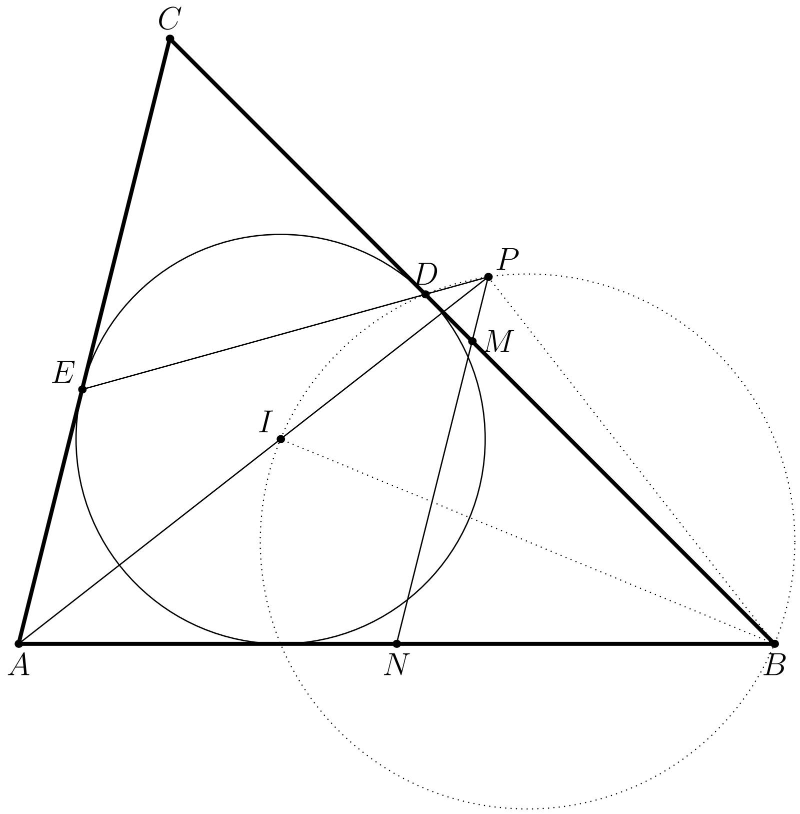

Figure 1: Problem 2

Problem 3. *Alice and Bob determine a number with* 2018 *digits in the decimal system by choosing digits from left to right. Alice starts and then they each choose a digit in turn. They have to observe the rule that each digit must differ from the previously chosen digit modulo* 3.

Since Bob will make the last move, he bets that he can make sure that the final number is divisible by 3. *Can Alice avoid that?*

(Richard Henner)

Solution. It is well-known that every number is congruent to its sum of digits in the decimal system modulo 3. It is therefore sufficient to consider the digits modulo 3. In particular, it is enough to only consider digits in $\{1, 2, 3\}$.

In the fourth move from the end, Alice makes sure that the sum of the digits is not divisible by 3 after her move. This is always possible because she has two options which cannot both lead to multiples of 3. After the next move, the sum of digits is congruent to some x modulo 3, but x is not the digit chosen by Bob because the sum of digits was not a multiple of 3 before Bob's move.

In the penultimate move, Alice can therefore choose x. After her move, the sum of digits is congruent to $2x \equiv -x$ modulo 3. In order to get a multiple of 3, Bob would have to choose another x, which is prohibited.

Therefore, Bob cannot reach his goal.

(Theresia Eisenkölbl) □

Problem 4. *Let M be a set containing positive integers with the following three properties:*

(1) $2018 \in M$.

(2) If $m \in M$, then all positive divisors of m are also elements of M.

(3) For all elements $k, m \in M$ with $1 < k < m$, the number $km + 1$ is also an element of M.

Prove that $M = \mathbb{Z}_{\geq 1}$.

(Walther Janous)

Solution. We first show that 1, 2, 3, 4, 5 are elements of M:

As divisors of 2018, the numbers 1, 2 and 1009 are elements of M. Therefore, $2019 = 2 \cdot 1009 + 1$ and its divisor 3 are elements of M. We now obtain $7 = 2 \cdot 3 + 1$ and $15 = 2 \cdot 7 + 1$ and therefore the divisor 5 of 15 as elements of M. Considering $16 = 3 \cdot 5 + 1$, we see that $4 \in M$.

We now show by induction that $\{1, 2, \ldots, 2k - 1\} \subseteq M$ for $k \geq 1$.

This has been shown above for $k \leq 3$. Assume that the assertion holds for some $k \geq 3$. Then we only have to verify that $2k$ and $2k + 1$ are elements of M, too.

It is clear that $2k + 1 = 2 \cdot k + 1$ is an element of M due to $k \geq 3$. This implies that $(2k)^2 = (2k - 1)(2k + 1) + 1$ and its divisor $2k$ are elements of M. This concludes the proof of the assertion and shows that M consists of all positive integers.

(Theresia Eisenkölbl) □