# $46^{\text{th}}$ Austrian Mathematical Olympiad

National Competition (Final Round, part 1)

May 1, 2015

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**Problem 1.** *Let $a$, $b$, $c$, $d$ be positive numbers. Prove that*

$$(a^2+b^2+c^2+d^2)^2 \geq (a+b)(b+c)(c+d)(d+a).$$

*When does equality hold?*

(Georg Anegg)

*Solution.* By the inequality between the arithmetic and the geometric mean, we have

$$(a+b)(b+c)(c+d)(d+a) \ \leq \ \left(\frac{(a+b)+(b+c)+(c+d)+(d+a)}{4}\right)^4 \ = \ 2^4\left(\frac{a+b+c+d}{4}\right)^4.$$

By the inequality beetween the quadratic and the arithmetic mean, we have

$$2^4\left(\frac{a+b+c+d}{4}\right)^4 \leq 2^4\left(\frac{a^2+b^2+c^2+d^2}{4}\right)^2 = (a^2+b^2+c^2+d^2)^2,$$

as required.

In the second inequality, equality holds if and only if $a=b=c=d$, but in that case, equality holds also in the original inequality. Therefore, equality holds if and only if $a=b=c=d$.

*(Clemens Heuberger)* □

**Problem 2.** *Let $ABC$ be an acute-angled triangle with $AC < AB$ and circumradius $R$. Furthermore, let $D$ be the foot of the altitude from $A$ on $BC$ and let $T$ denote the point on the line $AD$ such that $AT = 2R$ holds with $D$ lying between $A$ and $T$. Finally, let $S$ denote the mid-point of the arc $BC$ on the circumcircle that does not include $A$.*

*Prove: $\angle AST = 90\,^\circ$.*

(Karl Czakler)

*Solution.* As usual, we denote the angles $\angle BAC$, $\angle ABC$ and $BCA$ by $\alpha$, $\beta$ and $\gamma$, respectively. The center of the circumcircle is denoted by $O$, see Figure 1.

By assumption, we have $\beta < \gamma$. Let $E$ be the point on the circumcircle of $ABC$ diametrically opposite to $A$.

By the inscribed angle theorem, we have $\angle AOB = 2\gamma$. By definition, $S$ is the intersection of the angular bisector of $\angle CAB$ and the circumcircle. We note that

$$\angle EAS = \angle BAS - \angle BAO = \frac{\alpha}{2} - \frac{1}{2}(180^\circ - \angle AOB) = \frac{\alpha}{2} - \frac{1}{2}(180^\circ - 2\gamma) = \frac{\alpha}{2} + \gamma - 90^\circ -$$

Since we also have

$$\angle SAT = \angle SAC - \angle DAC = \frac{\alpha}{2} - (90^\circ - \angle ACD) = \frac{\alpha}{2} + \gamma - 90^\circ$$

it therefore follows that $\angle EAS = \angle TAS$ holds. Since we also have $AE = AT = 2R$, triangles $ASE$ and $AST$ are congruent, and therefore $\angle AST = \angle ASE$ follows. Since $AE$ is a diameter of the circumcircle, we have $\angle ASE = 90^\circ$, and the claim is proven.

*(Robert Geretschläger)* □

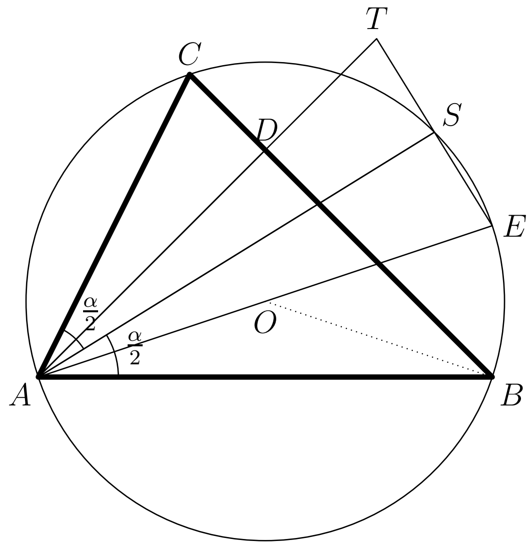


Figure 1: Problem 2

**Problem 3.** *Alice and Bob play a game with a string of 2015 pearls.*

*In each move, one player cuts the string between two pearls and the other player chooses one of the resulting parts of the string while the other part is discarded.*

*In the first move, Alice cuts the string, thereafter, the players take turns.*

*A player loses if he or she obtains a string with a single pearl such that no more cut is possible.*

*Who of the two players does have a winning strategy?*

(Theresia Eisenkölbl)

*Solution.* We claim that the winning situations are exactly the strings of an even number of pearls. We prove this claim by induction.

A string with one pearl is a losing situation by definition.

A string with an even number $n$ of pearls can easily be cut into two odd parts. These parts are a losing situation for the other player by induction, so that $n$ is a winning situation.

For an odd number of pearls, each cut produces an even part. The other player can thus choose this even part, which is a winning situation by induction. Therefore, the odd number $n$ is a losing position.

We conclude that Bob has a winning strategy.

*(Theresia Eisenkölbl)* □

**Problem 4.** *A* police emergency number *is a positive integer that ends with the digits* 133 *in decimal representation. Prove that every police emergency number has a prime factor larger than* 7.

*(In Austria,* 133 *is the emergency number of the police.)*

(Robert Geretschläger)

*Solution.* Let $n = 1000k + 133$ be a police emergency number and assume that all its prime divisors are at most 7. It is clear from the last digit that $n$ is odd and that $n$ is not divisible by 5, so $1000k + 133 = 3^a 7^b$ for suitable integers $a$, $b \geq 0$.

Thus $3^a 7^b \equiv 133 \pmod{1000}$.

This also implies $3^a 7^b \equiv 133 \equiv 5 \pmod 8$. We know that $3^a$ is congruent to 1 or 3 modulo 8 and $7^b$ is congruent to 1 or 7 modulo 8. In order for the product $3^a 7^b$ to be congruent to 5 modulo 8, $3^a$ must therefore be congruent to 3 and $7^b$ must be congruent to 7. We therefore conclude that $a$ and $b$ are both odd.

We also have $3^a 7^b \equiv 133 \equiv 3 \pmod 5$. As $a$ and $b$ are odd, $3^a$ and $7^b$ are each congruent to 3 or 2 modulo 5. Neither $3^2$, $2^2$ nor $3 \cdot 2$ is congruent to 3 modulo 5, a contradiction.

*(Clemens Heuberger)* □