57^{th} Austrian Mathematical Olympiad

Junior Regional Competition—Solutions

16th June 2026

Problem 1. *Let a, b, c be positive real numbers with $a+b \neq c$, $b+c \neq a$ and $c+a \neq b$.*
Prove that at least one of the three numbers

$$\frac{c}{a+b-c}, \quad \frac{a}{b+c-a}, \quad \frac{b}{c+a-b}$$

is less than or equal to 1.

(Karl Czakler)

Solution. We carry out a proof by contradiction. Suppose all three numbers are greater than 1, i.e.,

$$\frac{c}{a+b-c} > 1, \qquad \frac{a}{b+c-a} > 1, \qquad \frac{b}{c+a-b} > 1.$$

In particular, this means the three denominators are positive. Therefore, we can multiply by the denominators to obtain

$$c > a+b-c, \qquad a > b+c-a \qquad b > c+a-b.$$

Adding these three inequalities together yields

$$a+b+c > a+b+c.$$

This is a contradiction, which means our assumption must have been false and one of the three numbers must have been less than or equal to 1.

(Karl Czakler) □

Problem 2. *Let $ABCD$ be a trapezoid with $AB \parallel CD$, $AB > CD$ and $\angle BAD + \angle CBA = 90°$. Furthermore, let M and N be the midpoints of the sides AB and CD, respectively.*
Prove:

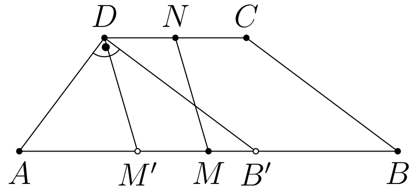
$$2 \cdot MN = AB - CD.$$

(Karl Czakler)

Solution. Let B' be the point on AB such that $B'D$ is parallel to BC. We have

$$\angle ADB' = 180° - (\angle B'AD + \angle DB'A) = 90°$$

and $AB' = AB - CD$. Let M' be the point on AB such that $M'D$ is parallel to MN.



It follows that

$$2 \cdot AM' = 2 \cdot (AM - MM') = AB - CD.$$

Using $AB' = AB - CD$, we obtain that M' is the midpoint of AB'. By Thales's Theorem, it follows that

$$2 \cdot MN = 2 \cdot M'D = 2 \cdot B'M' = 2 \cdot AM' = AB - CD.$$

(Karl Czakler) □

Problem 3. *Every tile in a game of dominoes is made of two halves. Each half displays one of the numbers* 0, 1, …, 6. *Every tile* $\boxed{a|b}$ *with* $0 \le a \le b \le 6$ *occurs exactly once. We consider* $\boxed{b|a}$ *to be the same tile as* $\boxed{a|b}$.

A row *is formed by placing at least two tiles after each other in such a way that the numbers on neighboring halves of consecutive tiles are always equal.*

A row is called closed *if the numbers on both ends of the row are also equal.*

For example: $\boxed{0|2}$, $\boxed{2|4}$, $\boxed{4|4}$, $\boxed{4|0}$ *is a closed row.*

Determine the maximum number of closed rows that can be formed simultaneously with this game of dominoes. (It is allowed to have some tiles left over.)

(Karl Czakler)

Solution. We start with some simple observations:

- A row consisting of two tiles cannot be closed since it would have to consist of two identical tiles $\boxed{a|b}$ and $\boxed{b|a}$. Every closed row must therefore consist of at least three tiles.
- If a closed row contains one ore several double tiles $\boxed{a|a}$, they can be removed and the row will remain closed.
- By removing all double tiles the number of closed rows is not changed. One can therefore assume that these tiles are not used.
- In total, there are $\binom{7}{2} = 21$ tiles, which are not double tiles.

Since we need at least 3 tiles for each closed row, one can form at most 7 closed rows from the 21 tiles. The following example shows that it is indeed possible to form 7 closed rows from the 21 tiles:

$$\begin{array}{l}
\boxed{0|1}, \boxed{1|2}, \boxed{2|0} \\
\boxed{0|3}, \boxed{3|4}, \boxed{4|0} \\
\boxed{0|5}, \boxed{5|6}, \boxed{6|0} \\
\boxed{1|3}, \boxed{3|5}, \boxed{5|1} \\
\boxed{1|4}, \boxed{4|6}, \boxed{6|1} \\
\boxed{2|3}, \boxed{3|6}, \boxed{6|2} \\
\boxed{2|4}, \boxed{4|5}, \boxed{5|2}
\end{array}$$

□

Problem 4. *Determine all positive integers* k *such that*

$$2^k + 3^k + 4^k$$

is a perfect square.

(Walther Janous)

Answer. The only positive integer k with this property is $k = 1$.

Solution. We denote the numbers under consideration by z_k.

- For $k = 1$, we obtain $z_1 = 2 + 3 + 4 = 3^2$, which is a perfect square.
- Let $k \geq 2$. Then 2^k is divisible by 4, and we have

  $$z_k \equiv 0 + (-1)^k + 0 \equiv 3^k \pmod{4}.$$

  Since 3 is not a quadratic residue modulo 4, z_k can be a perfect square only if k is even. However, if k is even, then

  $$z_k \equiv (-1)^k + 0 + 1^k \equiv 2 \pmod{3}.$$

  Since 2 is not a quadratic residue modulo 3, z_k is not a perfect square.

Therefore, $k = 1$ is the only positive integer for which $2^k + 3^k + 4^k$ is a perfect square. $\square$