47^{th} Austrian Mathematical Olympiad

Beginners' Competition – Solutions

June 16, 2016

Problem 1. *Determine all nonnegative integers n having two distinct positive divisors with the same distance from $\frac{n}{3}$.*

(Richard Henner)

Solution. Since the smallest possible divisors of an integer n are 1, 2 and 3, the greatest possible divisors are n, $\frac{n}{2}$ and $\frac{n}{3}$. Hence a divisor that is bigger than $\frac{n}{3}$ can only be n or $\frac{n}{2}$. Since there is no positive divisor of n having the same distance from $\frac{n}{3}$ as n, the bigger one of the two divisors must be $\frac{n}{2}$. The distance from $\frac{n}{2}$ to $\frac{n}{3}$ equals $\frac{n}{6}$ and since $\frac{n}{3} - \frac{n}{6} = \frac{n}{6}$ the smaller divisor must be $\frac{n}{6}$. Hence n is a multiple of 6. On the other hand it is clear that all positive multiples of 6 have the desired property.

(Gerhard Kirchner) □

Problem 2. *Prove that all real numbers $x \neq -1$, $y \neq -1$ with $xy = 1$ satisfy the following inequality:*

$$\left(\frac{2+x}{1+x}\right)^2 + \left(\frac{2+y}{1+y}\right)^2 \geq \frac{9}{2}$$

(Karl Czakler)

Solution. Since $xy = 1$, we may assume that $x \neq 0$ and $y \neq 0$. By substituting $y = \frac{1}{x}$ we achieve

$$\left(\frac{2+x}{1+x}\right)^2 + \left(\frac{2+y}{1+y}\right)^2 = \left(\frac{2+x}{1+x}\right)^2 + \left(\frac{2x+1}{x+1}\right)^2 = \frac{5x^2+8x+5}{x^2+2x+1},$$

and it remains to show that

$$\frac{5x^2+8x+5}{x^2+2x+1} \geq \frac{9}{2}.$$

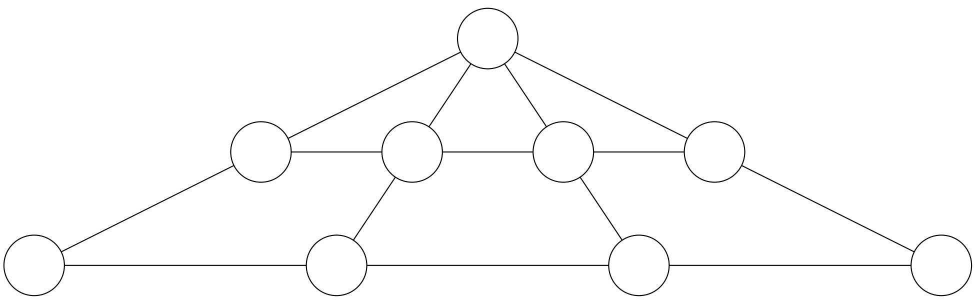
This inequality is equivalent to the inequality

$$(x-1)^2 \geq 0$$

and hence everything is proved.

(Karl Czakler) □

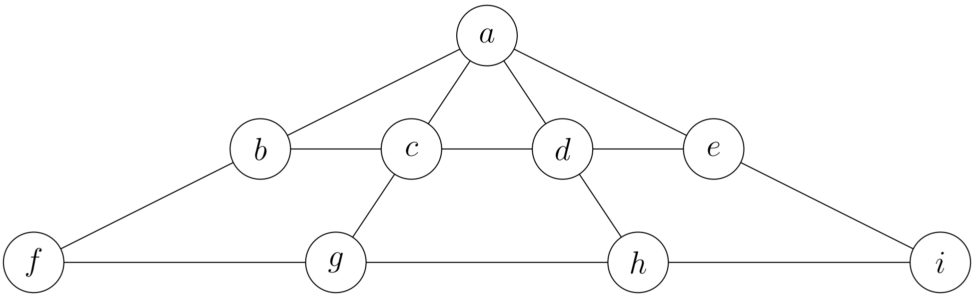
Problem 3. *We consider the following figure:*



We are looking for labellings of the nine fields with the numbers 1, 2, ..., 9. Each of these numbers has to be used exactly once. Moreover, the six sums of three resp. four numbers along the drawn lines have to be be equal.
Give one such labelling.
Show that all such labellings have the same number in the top field.
How many such labellings do there exist? (Two labellings are considered different, if they disagree in at least one field.)

(Walther Janous)

Solution. We denote the numbers in the fields of the figure by the following variables:



We denote the common value of the six sums by s and we get the following system of equations:

$$a+b+f=s, \tag{1}$$
$$a+c+g=s, \tag{2}$$
$$a+d+h=s, \tag{3}$$
$$a+e+i=s, \tag{4}$$
$$b+c+d+e=s, \tag{5}$$
$$f+g+h+i=s, \tag{6}$$
$$a+b+c+d+e+f+g+h+i=1+2+\ldots+9=45. \tag{7}$$

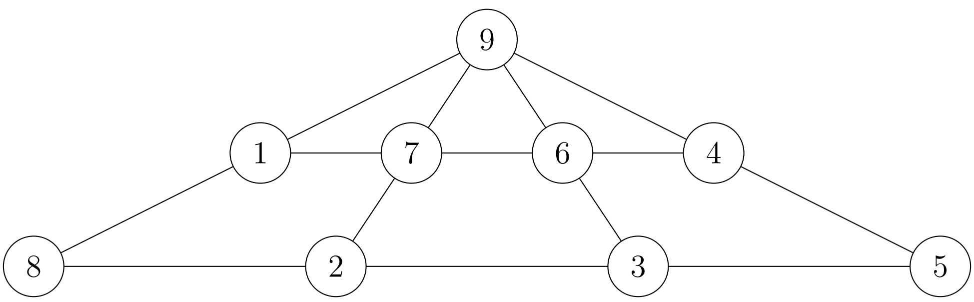
By adding the equations (1)–(4) and subtracting (5) and (6), we achieve $4a=2s$, i.e. $a=\frac{s}{2}$. To this we now add (5) and (6) and using (7) we get

$$\frac{s}{2}+s+s=a+b+c+d+e+f+g+h+i=45.$$

This means $s=18$ and $a=9$. Equations (1)–(6) now take the form

$$b+f=c+g=d+h=e+i=9, \tag{8}$$
$$b+c+d+e=f+g+h+i=18. \tag{9}$$

From (8) it follows that $\{\{b,f\},\{c,g\},\{d,h\},\{e,i\}\}=\{\{1,8\},\{2,7\},\{3,6\},\{4,5\}\}$.
If we search for solutions satisfying $\{b,f\}=\{1,8\}$, $\{c,g\}=\{2,7\}$, $\{d,h\}=\{3,6\}$, $\{e,i\}=\{4,5\}$, we see that equation (9) can be fulfilled neither with $b=1$, $c=2$ nor with $b=8$, $c=7$.
From these considerations we get the solutions $(a,b,c,d,e,f,g,h,i)=(9,1,7,6,4,8,2,3,5)$ and $(9,8,2,3,5,1,7,6,4)$. For instance we have the following labelling:



We have already shown that $a = 9$. Every other labelling can be obtained by interchanging (b, c, d, e) and (f, g, h, i) (2 possibilities) and independently permuting (b, f), (c, g), (d, h) and (e, i) (24 possibilities). So in total there exist 48 possible labellings.

(Gerhard Kirchner) $\square$

Problem 4. *Let $ABCDE$ be a convex pentagon with five equal sides and right angles at C and D. Let P denote the intersection point of the diagonals AC and BD.*
Prove that the segments PA and PD have the same length.

(Gottfried Perz)

Solution. $BCDE$ is a square since $\overline{BC} = \overline{CD} = \overline{DE}$ and $BC \perp CD$, $CD \perp DE$. Hence also the length of the segment $\overline{BE}$ coincides with the side length of the pentagon $ABCDE$ and we have $BC \perp BE$ and $BE \perp DE$. Furthermore $\overline{AB} = \overline{AE} = \overline{BE}$, hence ABE is an equilateral triangle. Now we have

$$\angle CBA = \angle CBE + \angle EBA = 90^\circ + 60^\circ = 150^\circ,$$
$$\angle AED = \angle AEB + \angle BED = 60^\circ + 90^\circ = 150^\circ.$$

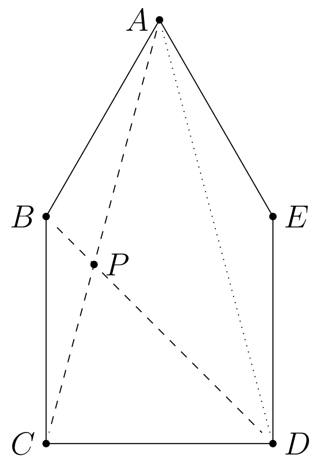
Since $\overline{AB} = \overline{BC} = \overline{DE} = \overline{EA}$ the isosceles triangles ABC and AED are congruent. We get

$$\angle BAC = \angle ACB = \angle DAE = \angle EDA = \frac{180^\circ - 150^\circ}{2} = 15^\circ.$$

Since every diagonal in a square bisects the right angles in its endpoints, we have

$$\angle ADP = \angle EDB - \angle EDA = 45^\circ - 15^\circ = 30^\circ,$$
$$\angle PAD = \angle BAE - \angle BAC - \angle DAE = 60^\circ - 2 \cdot 15^\circ = 30^\circ.$$

Hence ADP is an isosceles triangle with base AD and it follows that $\overline{PA} = \overline{PD}$.



(Gottfried Perz) $\square$