57$^{\text{th}}$ Austrian Mathematical Olympiad

Regional Competition—Solutions

26th March 2026

Problem 1. *Let a and b be positive real numbers with*

$$(a+4)(b+1) = 24.$$

Prove the inequality

$$a^2 + b^2 \geq 13.$$

When does equality hold?

(Karl Czakler)

Solution. Using the inequalities $a^2 + 4 \geq 4a$ and $b^2 + 9 \geq 6b$ (with equality when $a = 2$ and $b = 3$), we first obtain

$$\begin{aligned} a^2 + b^2 &= (a^2 + 4) + (b^2 + 9) - 13 \\ &\geq 4a + 6b - 13 \\ &= 4(a+4) + 6(b+1) - 35 \end{aligned}$$

Applying the AM-GM inequality (again with the equality case $a = 2$, $b = 3$: $4(2+4) = 6(3+1)$), we obtain

$$\begin{aligned} 4(a+4) + 6(b+1) - 35 &\geq 2\sqrt{24(a+4)(b+6)} - 35 \\ &= 2\sqrt{24^2} - 35 \\ &= 13. \end{aligned}$$

Summing up we have $a^2 + b^2 \geq 13$, which was to be proven.

(Karl Czakler) □

Problem 2. *Let k be the circumcircle of a square $ABCD$. Let P be a point on the shorter arc CD of k with $P \neq C$ and $P \neq D$. Let the intersection of the line BP with the line AC be Q, and let the intersection of the line CP with the line AD be R.*

Prove that the line RQ is perpendicular to the line AC .

(Karl Czakler)

Solution. Let H be the intersection point of AP and CD. Since $\angle APC = 90^0$, H is the orthocenter of the triangle ACR. Because

$$\angle HPQ = \angle APB = \angle ACB = 45^0 = \angle ACD$$

the quadrilateral $HQCP$ is a cyclic quadrilateral and therefore

$$90^0 = \angle APC = \angle HPC = \angle HQC.$$

It follows that Q is the foot of the altitude through the vertex R which completes the proof.

(Karl Czakler) □

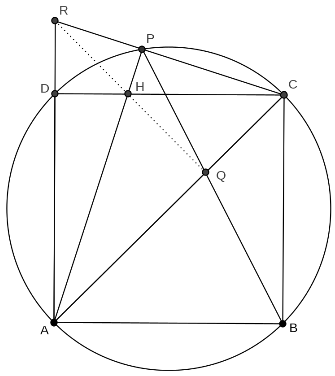

Figure 1: Problem 2

Problem 3. *Let n be a positive integer. A set of positive integers is called* airy, *if it does not contain two consecutive numbers. For each airy subset of the set $\{1, 2, \ldots, n\}$ compute the product of its elements, then add the squares of all these products where the empty subset is assigned the product 1.*

Determine the sum s_n.

Example: $n = 3$

The airy subsets of $\{1, 2, 3\}$ are $\{\}$, $\{1\}$, $\{2\}$, $\{3\}$ und $\{1, 3\}$. For the products, we get

$\{\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,3\}$
1	1	2	3	3

Therefore, the sum of the squares equals $s_3 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2 = 24$.

(Walther Janous)

Solution. First, we examine small cases of n.

- $s_1 = 1^2 + 1^2 = 2$
- $s_2 = 1^2 + 1^2 + 2^2 = 6$
- $s_3 = 1^2 + 1^2 + 2^2 + 3^2 + (1 \cdot 3)^2 = 24$
- $s_4 = 1^2 + 1^2 + 2^2 + 3^3 + 4^2 + (1 \cdot 3)^2 + (1 \cdot 4)^2 + (2 \cdot 4)^2 = 120$

If we compare these sum values with the corresponding values of $n!$, we arrive at the conjecture $s_n = (n+1)!$, which we will prove below using a two-step induction.

- The start of the induction can be found above (cases for $n = 1, 2, 3, 4$). Actually it would be sufficient to check the cases $n = 1$ and $n = 2$.
- Let us now assume that $s_{n-1} = n!$ and $s_n = (n+1)!$ are correct for some $n \geq 2$.
- We note that there are two possibilities for the subsets T of $\{1, 2, \ldots, n, n+1\}$, namely:
    - ∘ $n + 1 \notin T$. In this case we obtain the sum contribution s_n for s_{n+1}.
    - ∘ $n + 1 \in T$. Here $n \notin T$ and all other elements of T are in the set $\{1, 2, \ldots, n-1\}$. Therefore, we obtain the sum contribution $(n+1)^2 \cdot s_{n-1}$.

  Therefore, it follows that

$$s_{n+1} = s_n + (n+1)^2 s_{n-1},$$

  so according to the induction assumption

$$s_{n+1} = (n+1)! + (n+1)^2 n! = (n+1)!(n+2) = (n+2)!$$

  and the proof is complete.

(Walther Janous) □

Problem 4. *Prove the following two assertions.*

(a) There are infinitely many square numbers of the form $3^k + 3^n$ with positive integers k and n.

(b) There are no square numbers of the form $7^k + 7^n$ with positive integers k and n.

(Walther Janous)

Solution. (a) If we set $k = 2\ell$ even and $n = k + 1$, then

$$3^k + 3^n = 3^{2\ell} + 3^{2\ell+1} = (2 \cdot 3^\ell)^2,$$

so there are infinitely many solutions.

(b) With $7 \equiv 1 \pmod 3$, we obtain $7^k + 7^n \equiv 2 \pmod 3$. Since squares $\pmod 3$ can only have the remainders 0 or 1, the statement follows.

(Walther Janous) □