56^{th} Austrian Mathematical Olympiad

Regional Competition—Solutions

3rd April 2025

Problem 1. *Let $n \geq 3$ be a positive integer. Furthermore, let $x_1, \ldots, x_n \in [0, 2]$ be real numbers subject to $x_1 + \ldots + x_n = 5$.*

Prove the inequality

$$x_1^2 + \ldots + x_n^2 \leq 9.$$

When does equality hold?

(Walther Janous)

Answer. Equality holds if two numbers are 2, one number is 1 and the remaining numbers are 0.

Solution. We order the n numbers such that $x_1 \geq x_2 \geq \ldots \geq x_n$. Because of $9 = 4 + 5$ the inequality reads in equivalent form

$$\sum_{j=1}^{n} x_j^2 \leq 4 + \sum_{j=1}^{n} x_j,$$

that is $S \leq 4$ with

$$S = \sum_{j=1}^{n} x_j(x_j - 1).$$

The further considerations use facts of the parabola $p(x) = x(x-1)$ over the interval $[0, 2]$, in particular $p(x) = 2 \iff x = 2$, $p(x) = 0 \iff x \in \{0, 1\}$ and $p(x) < 0 \iff x \in (0, 1)$.

For $x_1 \leq 1$ we have $S \leq 0$. Let $k \geq 1$ be the index with $x_k > 1$ and $x_{k+1} \leq 1$. From $5 = \sum_{j=1}^{n} x_j \geq x_1 + \ldots + x_k > k$ we get $k \leq 4$.

- $k = 1$ leads to $S \leq 2$.
- $k = 2$ yields $S \leq 2 + 2 = 4$ with equality for $x_1 = x_2 = 2$ and $p(x_j) = 0$, $j = 3, \ldots, n$. This and $\sum_{j=1}^{n} x_j = 5$ imply $x_3 = 1$ and $x_4 = \ldots = x_n = 0$.
- For $k = 3$ we have $S \leq x_1(x_1 - 1) + x_2(x_2 - 1) + x_3(x_3 - 1)$. Consequently, $S \leq 2(x_1 - 1 + x_2 - 1 + x_3 - 1) \leq 2 \cdot (5 - 3) = 4$. In the first inequality, for equality one would have to have $x_1 = x_2 = x_3 = 2$, contradicting $x_1 + x_2 + x_3 \leq 5$.
- For $k = 4$ we arrive at a similar contradiction because $x_1 = x_2 = x_3 = x_4 = 2$ would now have to hold.

This proves the inequality. In it, equality holds if two numbers are 2, one number is 1 and the remaining numbers are 0 .

(Walther Janous) □

Problem 2. *Let ABC be an isosceles triangle with $AC = BC$ and circumcircle k. The line through B perpendicular to BC is denoted by n. Furthermore, let M be any point on n. The circle k_1 with center M and radius BM intersects AB once more at point P and the circumcircle k once more at point Q.*

Prove that the points P, Q and C lie on a straight line.

(Karl Czakler)

Solution. Let $\angle BAC = \angle CBA = \alpha$.

We consider the triangle PQB with the circumcircle k_1. By the chord tangent theorem we have

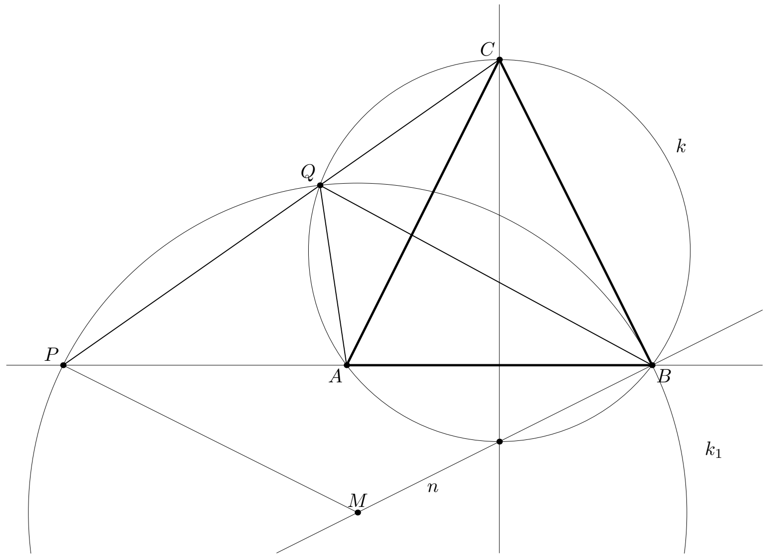
$$\angle BPQ = \angle CBQ$$

and therefore

$$\angle BPQ + \angle QBA = \angle CBQ + \angle QBA = \angle CBA = \alpha.$$

From the sum of the angles in the triangle PQB it follows that

$$\angle PQB = 180^0 - \alpha.$$



The triangle BQC has the circumcircle k and by the periphery angle theorem we obtain

$$\angle BQC = \angle BAC = \alpha.$$

The angles PQB and BQC are supplementary and therefore the points P, Q and C lie on a straight line.

(Karl Czakler) □

Problem 3. *There are* 6 *different bus lines in a city, each stopping at exactly* 5 *stations and running in both directions. Nevertheless, for every two different stations there is always a bus line connecting these two stations. Determine the maximum number of stations in this city.*

(Karl Czakler)

Answer.

Solution. In total, there are $6 \cdot 5 = 30$ stations. Every bus that stops at a particular station also stops at four other stations. Thus if there are more than $5 = 1 + 4$ stations, at least two lines must stop at each station. If there are more than $9 = 1+4+4$ stations, at least three lines must stop at each station. Let n be the number of stations. The following inequality holds

$$3n \leq 30,$$

thus

$$n \leq 10.$$

Hence the maximum number of stations is 10.

This number can actually be achieved: Denote the lines by $1, 2, \cdots 6$ and the stations by $A, B, \cdots J$. A possible timetable is given by:

Line	Station				
1	A	B	C	D	E
2	A	B	F	G	H
3	A	B	C	I	J
4	C	D	F	G	H
5	D	E	H	I	J
6	E	F	G	I	J

(Karl Czakler) □

Problem 4. *Let z be a positive integer that is not divisible by 8. Furthermore, let $n \geq 2$ be a positive integer.*

Prove that none of the numbers of the form $z^n + z + 1$ is a square number.

(Walther Janous)

Solution. We analyse the problem modulo 8 and consider the sequences $R(z) = (z^n + z + 1 \pmod 8)_{n \geq 2}$ for $z \equiv 1, 2, \ldots, 7$:

- For $z \equiv 1 \pmod 8$ we get $R(1) = (3, 3, 3, \ldots)$.
- For $z \equiv 2 \pmod 8$ we get $R(2) = (7, 3, 3, 3, \ldots)$.
- For $z \equiv 3 \pmod 8$ we get $R(3) = (5, 7, 5, 7, 5, 7, \ldots)$.
- For $z \equiv 4 \pmod 8$ we get $R(4) = (5, 5, 5, \ldots)$.
- For $z \equiv 5 \pmod 8$ we get $R(5) = (7, 3, 7, 3, 7, 3, \ldots)$.
- For $z \equiv 6 \pmod 8$ we get $R(6) = (3, 7, 7, 7, \ldots)$.
- For $z \equiv 7 \pmod 8$ we get $R(7) = (1, 7, 1, 7, 1, 7, \ldots)$.

Since the quadratic residues modulo 8 are the residues $0, 1, 4$, only the case $z \equiv 7 \pmod 8$ for even n remains to be analysed. In all other cases, $z^n + z + 1$ cannot be a square.

Let $n = 2k$ (with a positive integer k). But then

$$z^{2k} < z^{2k} + z + 1 < z^{2k} + 2z^k + 1 = (z^k + 1)^2,$$

so $z^n + z + 1$ is strictly between two consecutive square numbers and therefore cannot be a square number.

Therefore, $z^n + z + 1$ (for $n \geq 2$ and $n \not\equiv 0 \bmod 8$) is never a square number.

(Walther Janous) □