53^{rd} Austrian Mathematical Olympiad

Regional Competition—Solutions

31st March 2022

Problem 1. *Let a and b be positive real numbers with $a^2+b^2=\frac{1}{2}$. Prove that*

$$\frac{1}{1-a}+\frac{1}{1-b}\geq 4.$$

When does equality hold?

(Walther Janous)

Solution. Expansion and algebraic transformation lead to

$$3(a+b)\geq 2+4ab.$$

Squaring the inequality - the terms on both sides are postive - gives the equivalent inequality

$$9(a^2+b^2+2ab)\geq 4+16ab+16a^2b^2.$$

Via $a^2+b^2=\frac{1}{2}$, we obtain

$$\begin{array}{rrcl} & 9(\frac{1}{2}+2ab) & \geq & 4+16ab+16a^2b^2. \\ \Longleftrightarrow & 0 & \geq & 32a^2b^2-4ab-1 \end{array}$$

and therefore

$$0\geq (ab-\frac{1}{4})(32ab+4).$$

Because

$$ab\leq\frac{a^2+b^2}{2}=\frac{1}{4},$$

the first factor is less than or equal to 0. As the second factor is positive, the inequality is true, and equality holds for $a=b=\frac{1}{2}$.

(Karl Czakler) □

Problem 2. *Determine the number of ten-digit positive integers with the following properties:*

- *Each of the digits $0,1,2,\ldots,8$ and 9 is contained exactly once.*
- *Each digit, except 9, has a neighbouring digit that is larger than it.*

(Note. For example, in the number 1230, the digits 1 and 3 are the neighbouring digits of 2 and 2 and 0 are the neighbouring digits of 3. The digits 1 and 0 have only one neighbouring digit.)

(Karl Czakler)

Answer. There are 256 numbers with the required properties.

Solution. Let A be a ten-digit number with the desired properties. Let 10^k be the place value of the digit 9 with $k\in\{0,1,2\ldots,8,9\}$. All the digits of the number A to the left of 9 must be arranged in ascending order while all the digits to the right of 9 must be arranged in descending order. This implies that the digit 0 can only occur in the unit position and therefore the digit 9 cannot be in the unit position, i.e. $k>0$ holds.

We can distinguish the following cases:

- For $k = 9$, there is only one number, namely 9876543210.
- For $k = 8$, there are $\binom{8}{1} = 8$ numbers, as one of the remaining 8 digits must be chosen, which is placed left to 9. One example is the number 3987654210.
- For $k = 7$, there are $\binom{8}{2} = 28$ numbers, as there are $\binom{8}{2}$ possibilities of choosing 2 out of the remaining 8 digits, which are placed left to 9. Their order is already determined by this choice, as they must be arranged in ascending order. One example is the number 1498765320.
- According to the same consideration for $2 \le k \le 6$, there are $\binom{8}{9-k}$ possibilities (numbers) each.
- For $k = 1$, there exists only one ($1 = \binom{8}{8}$) number, namely 1234567890.

Accordingly, there exist a total of

$$\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \cdots + \binom{8}{8} = 2^8 = 256$$

numbers with the desired properties.

(Karl Czakler) □

Problem 3. *Let ABC denote a triangle with $AC \neq BC$. Let I and U denote the incenter and circumcenter of the triangle ABC, respectively. The incircle touches BC and AC in the points D and E, respectively. The circumcircles of the triangles ABC and CDE intersect in the two points C and P.*

Prove that the common point S of the lines CU and PI lies on the circumcircle of the triangle ABC.

(Karl Czakler)

Solution. Let S be the common point of CU and PI, see figure 1.

By Thales' theorem, we obtain that I lies on the circumcircle of the triangle CDE. Therefore, the following holds:

$$90° = \angle CDI = \angle CPI.$$

Thus, the triangle CPS is right-angled and we obtain

$$UC = UP = US.$$

Therefore, the point S lies on the circumcircle of the triangle ABC.

(Karl Czakler) □

Problem 4. *We are given the set*

$$M = \{-2^{2022}, -2^{2021}, \dots, -2^2, -2, -1, 1, 2, 2^2, \dots, 2^{2021}, 2^{2022}\}.$$

Let T be a subset of M, such that neighbouring numbers have the same difference when the elements are ordered by size.

(a) Determine the maximum number of elements that such a set T can contain.

(b) Determine all sets T with the maximum number of elements.

(Walther Janous)

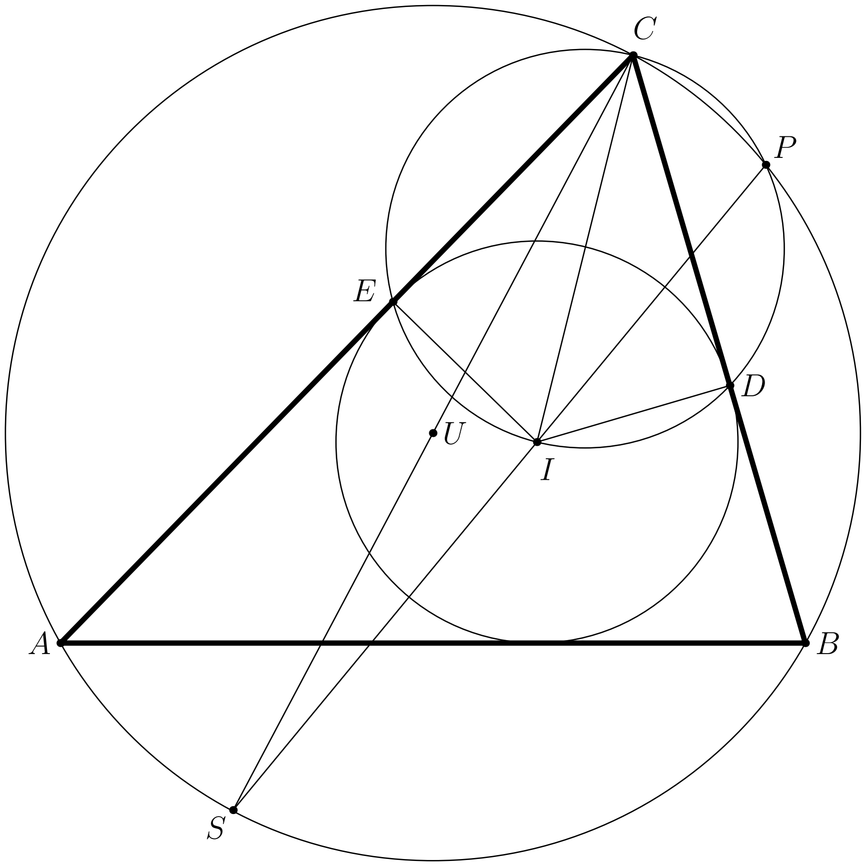

Figure 1: Problem 3

Solution. (a) First of all, we prove that a set T can contain at most two elements with the same sign. Assume that $2^a < 2^b < 2^c$ are three elements with the same (positive) sign. This implies both $a < b < c$ and

$$2^b - 2^a = 2^c - 2^b \iff 2^a(2^{b-a} - 1) = 2^b(2^{c-b} - 1).$$

Based on the unique prime factorisation, $a = b$ would therefore hold, which yields a contradiction. Because three negative numbers are also excluded with the same argument, there can never be more than two numbers in the set T with the same sign, i.e. a set T can contain at most four elements.

Let $-2^a < -2^b < 2^c < 2^d$ be the elements of T. In particular, $a > b$ and $c < d$ hold as well as

$$-2^b - (-2^a) = 2^c - (-2^b) \iff 2^a - 2^b = 2^c + 2^b \iff 2^a - 2^c = 2^{b+1}.$$

This equality implies that $a > c$. As $a > b$ (and equivalently $a \geq b+1$) we obtain that 2^{b+1} is a divisor of 2^a. Therefore, 2^{b+1} is also a divisor of 2^c. This implies $c \geq b+1$. Hence, the equation is equivalent to

$$2^{a-b-1} - 2^{c-b-1} = 1$$

with $a-b-1 > c-b-1 \geq 0$. If $c-b-1 > 0$ holds, the left side of the equation would be divisible by 2, which yields a contradiction. Therefore, $c = b+1$ holds and the equation is equivalent to

$$2^{a-b-1} - 1 = 1 \iff 2^{a-b-1} = 2 \iff a-b-1 = 1 \iff a = b+2.$$

We obtain that the elements of T have to be of the form $-2^{b+2} < -2^b < 2^{b+1} < 2^d$. But as furthermore

$$2^{b+1} - (-2^b) = 2^d - 2^{b+1} \iff 3 \cdot 2^b = 2^{b+1}(2^{d-b-1} - 1)$$

has to hold, this is a contradiction. In a similar way we obtain the following: If one assumes two positive numbers being in the set T, it can contain at most one negative number. This shows that 3 is the maximum number of elements that a set T can contain.

(b) It follows immediately that the sets T

- either consist of the three numbers -2^{b+2}, -2^b und 2^{b+1},
- or consist of the three numbers -2^{b+1}, 2^b und 2^{b+2},

with $0 \leq b \leq 2020$.

(Walther Janous) $\square$