48^{th} Austrian Mathematical Olympiad

Regional Competition (Qualifying Round)—Solutions

30th March 2017

Problem 1. *Let $x_1, x_2, \dots, x_9$ be nonnegative real numbers satisfying*

$$x_1^2 + x_2^2 + \dots + x_9^2 \ge 25.$$

Prove that there exist three of these numbers with a sum of at least 5.

(Karl Czakler)

Solution. (Karl Czakler) W.l.o.g. we may assume that $x_1 \ge x_2 \ge x_3 \ge x_4 \ge x_5 \ge x_6 \ge x_7 \ge x_8 \ge x_9 \ge 0$. Then it follows that $x_1x_2 \ge x_4^2 \ge x_5^2$, $x_1x_3 \ge x_6^2 \ge x_7^2$ and $x_2x_3 \ge x_8^2 \ge x_9^2$. Hence we have

$$(x_1+x_2+x_3)^2 = x_1^2+x_2^2+x_3^2+2x_1x_2+2x_1x_3+2x_2x_3 \ge x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+x_7^2+x_8^2+x_9^2 \ge 25.$$

Now $x_1 + x_2 + x_3 \ge 5$, which proves the assertion. □

Problem 2. *Let $ABCD$ be a cyclic quadrilateral with perpendicular diagonals and circumcenter O. Let g be the line obtained by reflection of the diagonal AC about the angle bisector of $\angle BAD$.*
Prove that the point O lies on the line g.

(Theresia Eisenkölbl)

Solution. (Gerhard Kirchner) Denote by X the point of intersection of the diagonals AC and BD, i. e. AX is an altitude in the triangle ABD, see Figure 1. By the inscribed angle theorem, we have

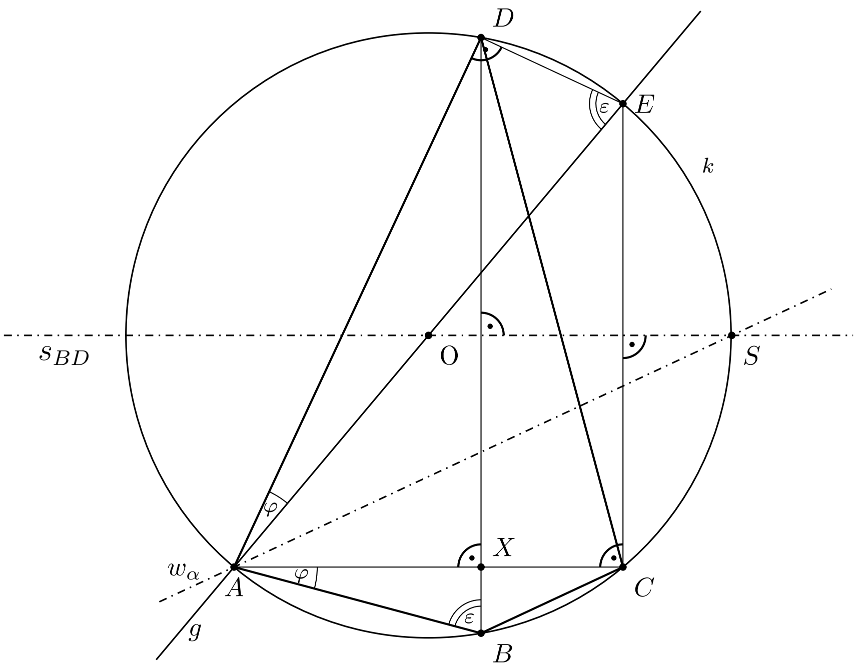

Figure 1: Problem 2

$\angle ABX = \frac{1}{2}\angle DOA$. Hence

$$\angle XAB = 90° - \angle ABX = \frac{1}{2} \cdot (180° - \angle DOA) = \angle OAD.$$

In the last step the angle sum in the equilateral triangle DAU has been used. Since the lines AB and AD are symmetric with respect to the angle bisector w_α, the same is true for AX und AU. Hence the assertion follows. □

Problem 3. *The nonnegative integers 2000, 17 and n are written on a blackboard. Alice and Bob play the following game: Alice begins, then they play in turns. A move consists in replacing one of the three numbers by the absolute difference of the other two. No moves are allowed where all three numbers remain unchanged. A player in turn who cannot make a legal move loses the game.*

- *Prove that the game will end for every number n.*
- *Who wins the game in the case $n = 2017$?*

(Richard Henner)

Solution. (Richard Henner) If three numbers are written on the blackboard and one of them is replaced by the (positive) difference of the other two, then after this move one number on the blackboard will be the sum of the other two. Let a, b and $a+b$ be the numbers on the blackboard; w.l.o.g. we assume that $b > a$. Because of $a+b-b = a$ and $a+b-a = b$ there is only one possible move. After it the numbers a, b and $b-a$ are written on the blackboard. Again, one number (namely b) is the sum of the other two and there exists only one possible move.
This means that at the latest from the second turn on there is no choice of moves and all moves are inevitable. Furthermore, from the second move on, the largest of the three numbers is decreased, and since no number can become negative, after a finite number of moves one of the numbers will be 0. Since 0 is the difference of the other two numbers, we must have 0, a, a on the blackboard. Now $a-0 = a$ and $a-a = 0$, therefore no further move is possible. Thus the player writing 0, a, a onto the blackboard is the winner.
If the game starts with the numbers 2000, 17 and 2017 on the blackboard, the course of the game is as follows:

$$\begin{aligned}
1^{\text{st}} &\text{ move (A): } 2000,\ 17,\ 1983\\
2^{\text{nd}} &\text{ move (B): } 1966,\ 17,\ 1983\\
3^{\text{rd}} &\text{ move (A): } 1966,\ 17,\ 1949\\
&\text{ etc. (since } 2000 : 17 = 117.6\ldots)\\
117^{\text{th}} &\text{ move (A): } 2000 - 116 \cdot 17 = 28,\ 17,\ 2000 - 117 \cdot 17 = 11\\
118^{\text{th}} &\text{ move (B): } 6,\ 17,\ 11\\
119^{\text{th}} &\text{ move (A): } 6,\ 5,\ 11\\
120^{\text{th}} &\text{ move (B): } 6,\ 5,\ 1\\
121^{\text{st}} &\text{ move (A): } 4,\ 5,\ 1\\
122^{\text{nd}} &\text{ move (B): } 4,\ 3,\ 1\\
123^{\text{rd}} &\text{ move (A): } 2,\ 3,\ 1\\
124^{\text{th}} &\text{ move (B): } 2,\ 1,\ 1\\
125^{\text{th}} &\text{ move (A): } 0,\ 1,\ 1
\end{aligned}$$

and A wins the game. □

Problem 4. *Determine all integers $n \geq 2$ that have a representation*

$$n = a^2 + b^2,$$

where a is the smallest divisor of n different from 1 and b is an arbitrary divisor of n.

(Walther Janous)

Solution. (Clemens Heuberger) If n is odd, then both a and b are odd and therefore $n = a^2 + b^2$ is even, contradiction. Therefore, n is even and $a = 2$. This also shows that b is even. Furthermore, $b \mid (n - b^2) = a^2 = 4$. Thus $b \in \{2, 4\}$, which results in $n = 8$ and $n = 20$, respectively. □