47^{th} Austrian Mathematical Olympiad

Regional Competition (Qualifying Round)

March 31, 2016

Problem 1. *Determine all positive integers k and n satisfying the equation*

$$k^2 - 2016 = 3^n.$$

(Stephan Wagner)

Solution. We immediately see that $n = 1$ does not lead to a solution, while $n = 2$ yields the solution $(k, n) = (45, 2)$.

We show that there is no solution with $n \geq 3$. In that case 3^n is divisible by 9 and thus k^2 is divisible by 9 which implies that $k = 3\ell$ for some positive integer ℓ. After division by 9, the equation reads $\ell^2 - 224 = 3^{n-2}$. Modulo 3, this yields $\ell^2 - 2 \equiv 0 \pmod 3$, a contradiction because 2 is not a quadratic residue modulo 3. *(Clemens Heuberger)* □

Problem 2. *Let a, b, c and d be real numbers with $a^2 + b^2 + c^2 + d^2 = 4$.*
Prove that the inequality

$$(a + 2)(b + 2) \geq cd$$

holds and give four numbers a, b, c and d such that equality holds. (Walther Janous)

Solution. The affirmed inequality is equivalent to $2ab + 4a + 4b + 8 \geq 2cd$, which can be written as

$$2ab + 4a + 4b + a^2 + b^2 + c^2 + d^2 + 4 \geq 2cd$$

on account of the condition $a^2 + b^2 + c^2 + d^2 = 4$. By the identity

$$a^2 + b^2 + 2ab + 4a + 4b + 4 = (a + b + 2)^2$$

we arrive at the equivalent and obvious inequality

$$(a + b + 2)^2 + (c - d)^2 \geq 0.$$

The case of equality occurs for

$$a + b = -2 \qquad \text{and} \qquad c = d \qquad \text{together with} \qquad a^2 + b^2 + c^2 + d^2 = 4,$$

for instance when $a = b = c = d = -1$. *(Walther Janous)* □

Problem 3. *On the occasion of the 47^{th} Mathematical Olympiad 2016 the numbers 47 and 2016 are written on the blackboard. Alice and Bob play the following game: Alice begins and in turns they choose two numbers a and b with $a > b$ written on the blackboard, whose difference $a - b$ is not yet written on the blackboard and write this difference additionally on the board. The game ends when no further move is possible. The winner is the player who made the last move.*
Prove that Bob wins, no matter how they play. (Richard Henner)

Solution. We consider the set B of the numbers on the blackboard at the end of the game. It is clear that $B \subseteq \{1, \ldots, 2016\}$. Let $m = \min B$ and $n \in B$. We claim that $m \mid n$. Otherwise, write $n = qm + r$ with $0 < r < m$. By induction on k, we have $n - km \in B$ for $0 \leq k \leq q$ (because no more moves are possible, these numbers must be on the blackboard). Thus $r = n - qm \in B$, which contradicts the minimality of m.

We conclude that $m \mid 1 = \gcd(2016, 47) \in B$. By induction on ℓ, we have $n - \ell \in B$ for $0 \leq \ell \leq 2015$. This also implies that $B = \{1, \ldots, 2016\}$.

As 2 numbers had been on the blackboard at the beginning of the game, the game ends after 2014 moves when all other numbers have been written. Therefore, Bob wins after move 2014.

(Clemens Heuberger) □

Problem 4. *Let ABC be a triangle with $AC > AB$ and circumcenter O. The tangents to the circumcircle at A and B intersect at T. The perpendicular bisector of the side BC intersects side AC at S.*

(a) Prove that the points A, B, O, S and T lie on a common circle.

(b) Prove that the line ST is parallel to the side BC.

(Karl Czakler)

Solution. Since AT and BT are perpendicular to AO and BO, the points A, B, T and O lie on a circle k_1 by Thales' theorem. By the central angle theorem we have $\angle AOB = 2\gamma$. Since BCS is an isoceles triangle, we find $\angle BCS = \angle CBS = \gamma$. Now $\angle ASB = 2\gamma$, because an exterior angle of a triangle equals the sum of the other two interior angles. Thus

$$\angle ASB = 2\gamma = \angle AOB,$$

and by the inscribed angle theorem we find that the points A, B, S und O lie on a circle k_2. Since the circles k_1 and k_2 have the three points A, B and O in common, we have $k_1 = k_2$ and the points A, B, O, S and T lie on a circle.
Finally we have $\angle TSB = \angle TOB = \gamma = \angle SBC$, from which it follows that ST is parallel to BC.

(Karl Czakler) □

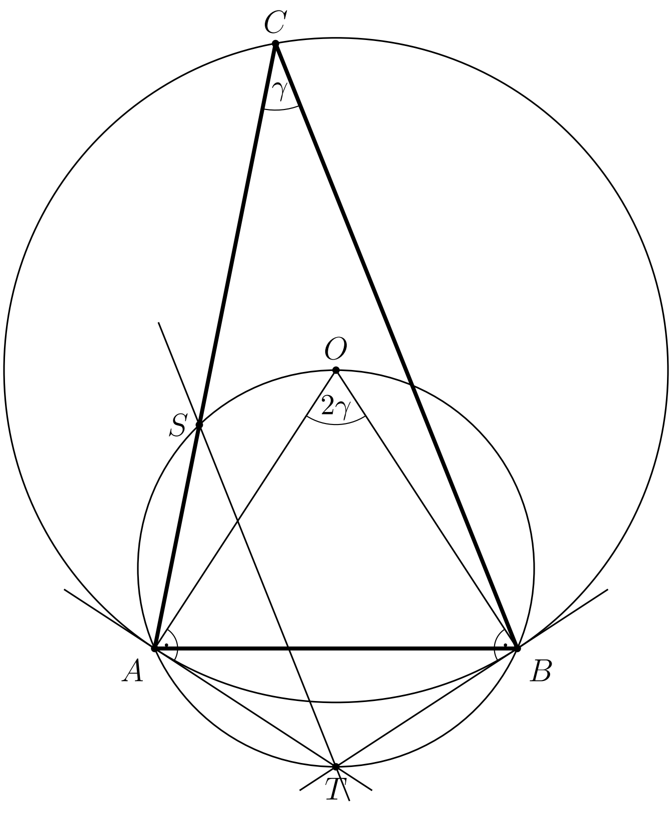

Figure 1: Problem 4