54^{th} Austrian Mathematical Olympiad

National Competition—Final Round—Solutions

24th/25th May 2023

Problem 1. *Let α be a nonzero real number.*

Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ with

$$f(f(x+y)) = f(x+y) + f(x)f(y) + \alpha xy$$

for all x, $y \in \mathbb{R}$.

(Walther Janous)

Answer. For $\alpha = -1$, the identity is the only solution. For other values of α, there is no solution.

Solution. The functional equation immediately implies that f cannot be a constant function, as αxy would then have to be constant. In the following, we let (F) denote the given functional equation.

Setting $y = 1$, (F) gives us

$$f(f(x+1)) = f(x+1) + f(x)f(1) + \alpha x. \tag{1}$$

For $x = 1$ we therefore have

$$f(f(2)) = f(2) + f(1)^2 + \alpha. \tag{2}$$

and replacing x by $x+1$ then yields

$$f(f(x+2)) = f(x+2) + f(x+1)f(1) + \alpha(x+1). \tag{3}$$

For $y = 2$, (F) yields

$$f(f(x+2)) = f(x+2) + f(x)f(2) + 2\alpha x. \tag{4}$$

For $x = 0$, we therefore obtain

$$f(f(2)) = f(2) + f(0)f(2).$$

Together with (2) this gives us

$$f(0)f(2) = f(1)^2 + \alpha. \tag{5}$$

If we now take (F) and let $y = 0$ and replace x by $x+1$, we obtain

$$f(f(x+1)) = f(x+1) + f(x+1)f(0). \tag{6}$$

From (1) and (6) we have

$$f(x+1)f(0) = f(x)f(1) + \alpha x \tag{7}$$

and from (3) and (4)

$$f(x+1)f(1) = f(x)f(2) + \alpha x - \alpha. \tag{8}$$

If we multiply (7) by $f(2)$ and (8) by $f(1)$, we obtain

$$f(x+1)f(0)f(2) = f(x)f(1)f(2) + \alpha f(2)x$$

or

$$f(x+1)f(1)^2 = f(x)f(1)f(2) + \alpha f(1)x - \alpha f(1).$$

After subtracting and taking (5) into consideration, we therefore have

$$\alpha f(x+1) = \alpha(f(2) - f(1))x + \alpha f(1),$$

and thus (since $\alpha \neq 0$)

$$f(x+1) = (f(2) - f(1))x + f(1).$$

We see that f is a linear function, and $f(x) = ax + b$ with $a \neq 0$. Substitution then gives us

$$a^2x + a^2y + ab + b = ax + ay + b + a^2xy + abx + aby + b^2 + \alpha xy.$$

For $y = 0$ we obtain

$$a^2x + ab = (a + ab)x + b^2, \ x \in \mathbb{R},$$

an therefore by comparing coefficients $a^2 = a + ab$, or $a = 1 + b$, and $ab = b^2$. We therefore have $(1 + b)b = b^2$, and thus $b = 0$, and $a = 1$. For the only possible function $f(x) = x$, we obtain from (F) that $(1 + \alpha)xy = 0$, $x, y \in \mathbb{R}$, or $\alpha = -1$ must hold.

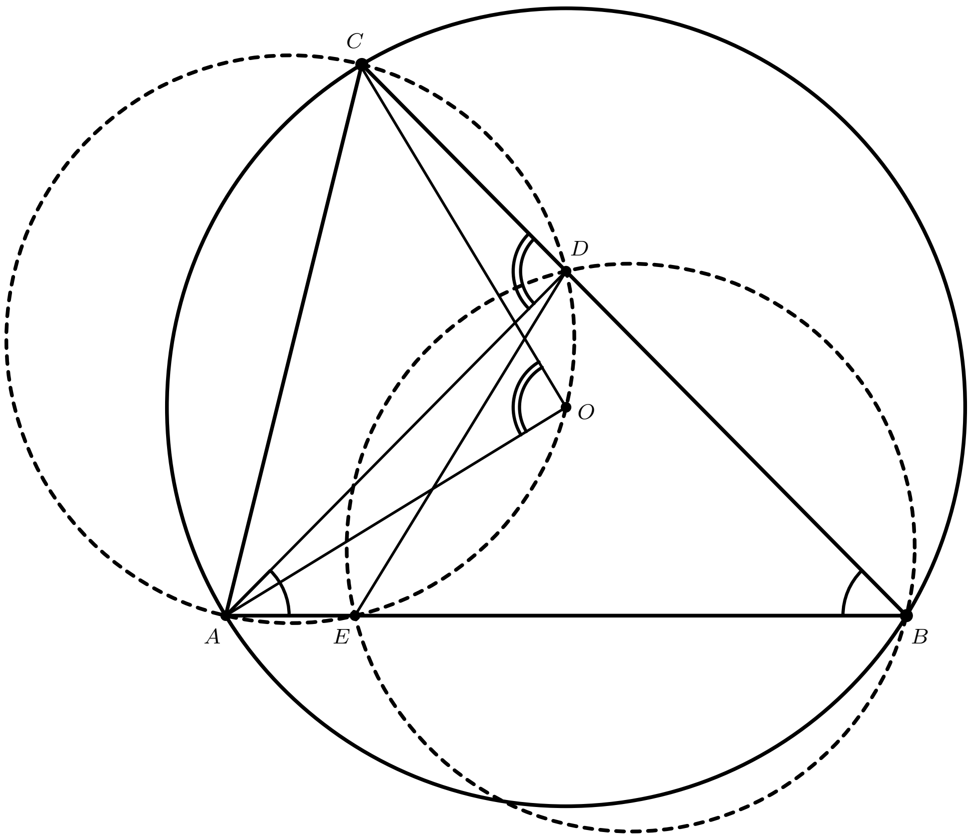
(Walther Janous) □

Problem 2. *Let ABC be a triangle, and O its circumcenter. The circumcircle of triangle AOC shall intersect the segment BC in points C and D and the segment AB in points A and E.*

Prove that triangles BDE and AOC have equal circumradii.

(Karl Czakler)

Solution. In the circumcircle of triangle ABC we have $\angle COA = 2\angle CBA$. In the circumcircle of ADC we therefore have $\angle CDA = \angle COA = 2\angle CBA$. The angle $\angle CDA$ is an external angle in triangle ABD, and we therefore obtain $\angle CBA + \angle BAD = \angle CDA = 2\angle CBA$, and thus $\angle BAD = \angle CBA$. In the circumcircle of triangle AOC we obtain $\angle BAD = \angle EAD$ on the chord ED. The angles $\angle CBA = \angle DBE$ are equal in the circumcircle of triangle BDE on the same chord ED. Since the chords and subtended angles are equal in both circles, they must have the same radii, as claimed.



(Theresia Eisenkölbl) □

Problem 3. *Alice and Bob play a game, in which they take turns drawing segments of length 1 in the Euclidean plane. Alice begins, drawing the first segment, and from then on, each segment must start at the endpoint of the previous segment. It is not permitted to draw the segment lying over the preceding one. If the new segment shares at least one point - except for its starting point - with one of the previously drawn segments, one has lost.*

a) Show that both Alice and Bob could force the game to end, if they don't care who wins.

b) Is there a winning strategy for one of them?

(Michael Reitmeir)

Solution. a) In the following, let A_n denote the end-point of the segment that Alice drew in her n-th turn (assuming the game has not ended by then), and let B_n denote the end-point of Bob's n-th segment. Furthermore, let B_0 denote the starting point of Bob's first segment.

If Alice can force an end to the game, so can Bob by applying the same strategy and ignoring Alice's first move. It is therefore sufficient to prove that Alice can force an end.

Bob must always choose the n-th end-point B_n on the circle with radius 1 and center in A_n. We name this circle k_n. Furthermore, let l_n denote the line perpendicular to $B_{n-1}A_n$ through A_n. We now note that if Bob chooses his end-point in such a way that his segment forms an acute angle with the preceding segment (such that B_n lies on the same side of l_n as the segment $\overline{B_{n-1}A_n}$), Alice can end the game with her next move. Now let h_n denote the part of k_n on the opposite side of l_n from $\overline{B_{n-1}A_n}$ (including the intersection points of l_n and k_n). In the following, we only need to consider the case in which Bob chooses the point B_n on the semi-circle h_n.

Let B denote the set of all points, whose distance from the first drawn segment is less than 1. The set B consists of a 1×2 rectangle and the interior of two semi-circles. Bob chooses B_1 on h_1. In the next move, Alice can choose A_2 as close as she wishes to A_1. Let r denote the distance between A_2 and A_1. We now consider two cases.

Case 1: B_0, A_1, B_1 do not lie on a common line.

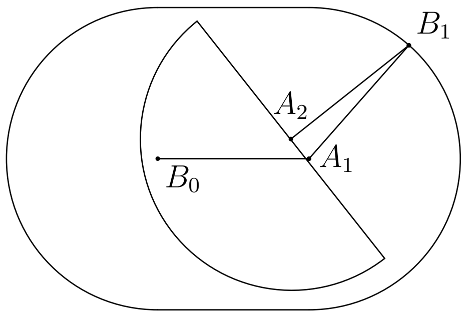


Figure 1: Problem 3

If Alice chooses $A_2 = A_1$ (which she is not allowed to do, according to the rules), h_2 will overlap with the semicircular edge of B at one end. The other end of h_2 must therefore lie in the interior of the rectangular section of B, which means that this end must have a positive distance from the edge of B. Since Alice can choose an arbitrarily small value of r, she can (for reasons of continuity) move the point A_2 away slightly from A_1 towards the rectangular section of B such that h_2 comes to lie completely in the interior of B. This means that B_2 will lie completely in the interior of B, and all its points thus have a distance less than 1 from the first segment. Alice can therefore certainly choose her next segment in such a way that it intersects the first segment.

Fall 2: B_0, A_1, B_1 lie on a common line.

In this case, Alice cannot choose A_2 in such a way that h_2 lies completely in the interior of B. If Bob

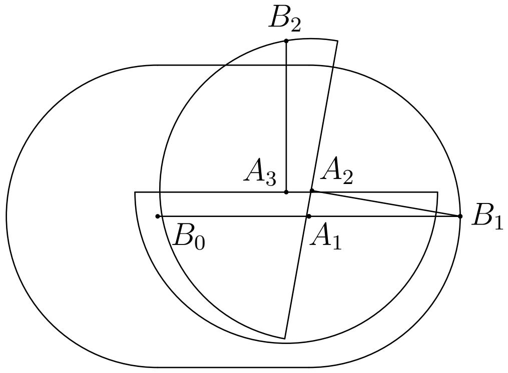


Figure 2: Problem 3

chooses B_2 in the interior of B, Alice can choose her next segement in such a way that it intersects the first segment, ending the game. We can therefore assume that Bob chooses B_2 on h_2 outside of B. In this case, Alice can choose her next point A_3 in such a way that its distance from A_2 is at most r. By the triangle inequality, the distance from A_3 to A_1 is then at most $2r$. If $r = 0$ (which is not allowed by the rules), we would have $A_3 = A_1$. In this case, analogously to the previous case, h_3 would overlap with the semicircular edge of B, and the other end would lie in the interior of the rectangular part of B with a positive distance from the edge. Since Alice can choose $2r$ arbitrarily small, she can (again by reasons of continuity) move A_3 slightly away from A_1 toward the part of h_3 in the interior of the rectangular section of B, such that h_3 comes to lie completely in the interior of B. Then B_3 lies in the interior of B, and Alice can choose her next segment in such a way that it intersects the first segment.

b) We will show that each of the players can always make a move with which they do not lose. This is trivially the case for the first two moves, so we assume without loss of generality that at least two segments have already been drawn. Let s denote the last segment drawn and t the one drawn immediately before that. Furthermore, let S denote the union of all segments that were drawn before s and t. Let r denote the smallest distance between any of the points of s and S. Since s and S are assumed to not have any common points, we certainly have $r > 0$.

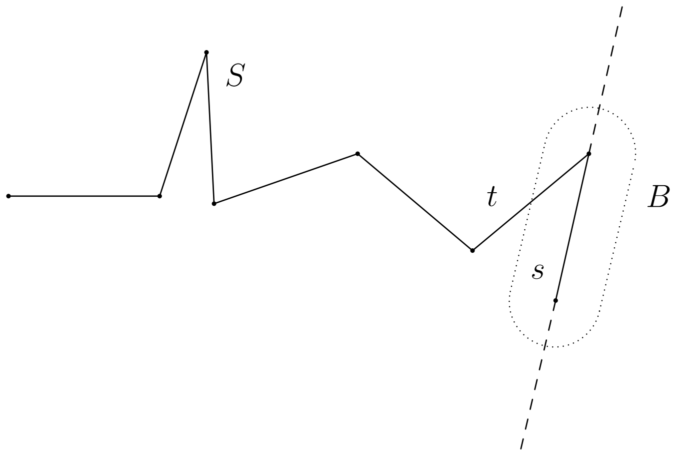


Figure 3: Problem 3

Now let B denote the set of all points x, whose distance from s ist less than $\frac{r}{2}$. B certainly does not contain any point from S. The only segments among those that have been drawn to this point that contain any of the points in B are thus s and t. Extending s to a line, we divide the euclidean plane into two half-planes, one of which certainly does not include any of the points of t. We choose this half-plane and determine its intersection with B. We can certainly find a segment of length 1 in this part of B, with one end in the end of s, that does not intersect either t or any of the other segments.

(Michael Reitmeir, Thomas Speckhofer) □

Problem 4. *Written on a blackboard are the* 2023 *numbers*

$$2023, 2023, \ldots, 2023.$$

The numbers on the blackboard are now modified, in a sequence of moves. In each move, two numbers on the blackboard—call them x and y—are chosen, deleted, and replaced by the single number $\frac{x+y}{4}$. Such moves are carried out until there is only one number left on the blackboard.

Prove that this number is always greater than 1.

(Walther Janous)

Solution. The expression $\frac{x+y}{4}$ reminds us of the arithmetic mean. By the AM-HM inequality, we have

$$\frac{x+y}{2} \geq \frac{2}{\frac{1}{x}+\frac{1}{y}},$$

or

$$\frac{1}{x}+\frac{1}{y} \geq \frac{1}{(x+y)/4}.$$

This inequality leads us to consider an argument concerning the reciprocals of the numbers on the board, as the sum of the reciprocals of two of these numbers is at least as large as the reciprocal of the number replacing them. This value remains the same if and only if the two chosen numbers are equal, and is otherwise larger. At the beginning, the sum of all reciprocals is

$$\frac{1}{2023}+\frac{1}{2023}+\ldots+\frac{1}{2023}=\frac{2023}{2023}=1.$$

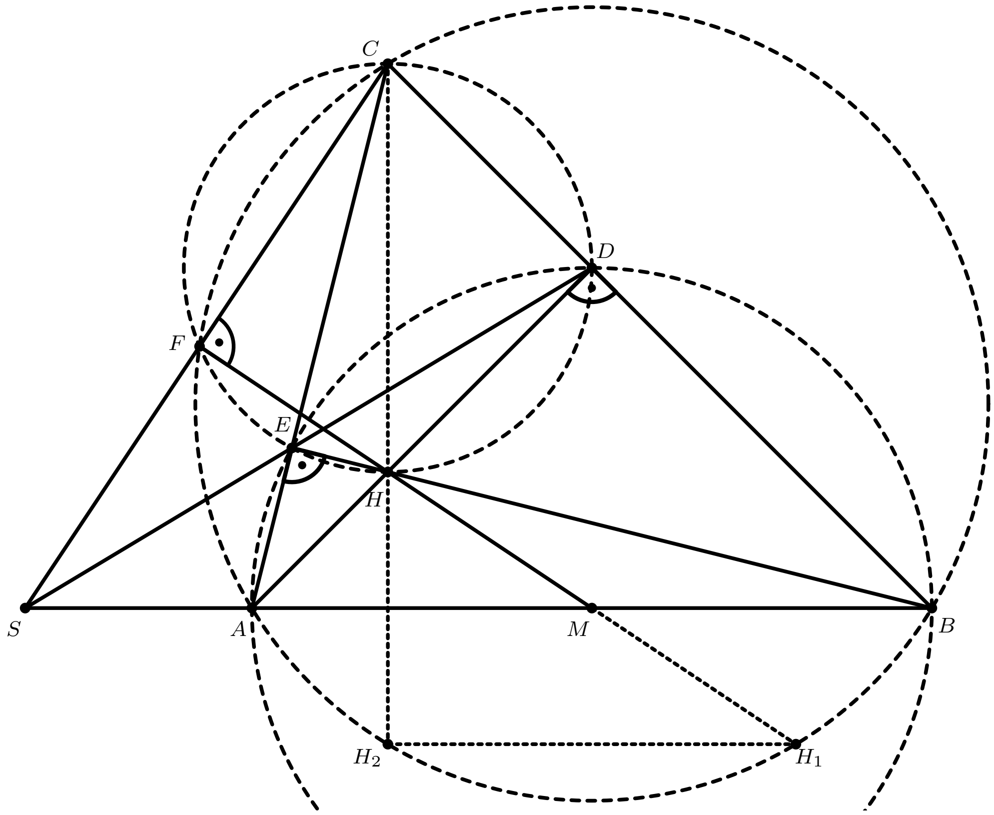
This implies the claim, since there is an odd number of 2023s in the beginning that cannot be divided into pairs, so one of them has to be part of a pair with different numbers.

(Walther Janous) □

Problem 5. *Let ABC be an acute triangle, with $AC \neq BC$. Let M be the midpoint of segment AB. Let H be the orthocenter of triangle ABC, D the footpoint of the altitude through A on BC and E the footpoint of the altitude through B on AC.*

Prove that lines AB, DE and the orthogonal to MH through C intersect in a point S.

(Karl Czakler)



Solution.

Let $\angle ACB = \gamma$ and F be the foot of C on MH. We will first demonstrate that F lies on the circumcircle k of triangle ABC.

Let H_1 denote the symmetric point to H with respect to M. The quadrilateral AH_1BH is a parallelogram, and since we have $\angle AHB = \angle AH_1B = 180° - \gamma$, the point H_1 must lie on the circumcircle k of ABC. Reflecting point H on triangle side AB yields point H_2, and it is well known that this point also lies on k. The line H_1H_2 is parallel to AB, and thus perpendicular to CH_2. It follows that CH_1 is a diameter of the circumcircle k, and it follows that F lies on k.
In summary, we have:

- Points A, B, D, E lie on a common circle k_1.
- Points C, E, H, D, F lie on a common circle k_2.
- Points A, B, F, C lie on the circumcircle k.

The point S is thus the radical center of these three circles, completing the proof.

(Karl Czakler, Josef Greilhuber) □

Problem 6. *Determine whether there exists a real number r such that the equation*

$$x^3 - 2023x^2 - 2023x + r = 0$$

has three different rational solutions.

(Walther Janous)

Solution. Let $N = 2023$. We assume that the equation $x^3 - Nx^2 - Nx + r = 0$ has three rational solutions $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}$, where a, b, c are integers and k is a positive integer with $\gcd(a, b, c, k) = 1$. According to Vieta we have $\frac{a}{k} + \frac{b}{k} + \frac{c}{k} = N$ and $\frac{b}{k} \cdot \frac{c}{k} + \frac{a}{k} \cdot \frac{c}{k} + \frac{a}{k} \cdot \frac{b}{k} = -N$. This is equivalent to

$$\begin{aligned} a + b + c = kN \quad &\Rightarrow \quad a^2 + b^2 + c^2 + 2(bc + ac + ab) = k^2N^2 \\ bc + ac + ab = -k^2N \quad &\Rightarrow \quad a^2 + b^2 + c^2 = k^2N^2 + 2k^2N = k^2N(N + 2). \end{aligned}$$

In a next step, we recognize that k cannot be even. If it were, we would have $a^2+b^2+c^2 \equiv 0 \mod 4$, from which we obtain that a, b, c are all even, as 0 and 1 are the only quadratic residues modulo 4. This contradicts the assumption that $\gcd(a, b, c, k) = 1$.

For odd values of k, we have $k^2 \equiv 1 \mod 8$. Furthermore, we have $N = 2023 \equiv 7 \mod 8$. From this, we obtain $k^2N(N+2) \equiv 1 \cdot 7 \cdot 1 \equiv 7 \mod 8$. The sum of three perfect squares can never be congruent to 7 modulo 8, which can easily be verified by adding all possible combinations (the only quadratic residues modulo 8 are $0, 1$ and 4). It follows that the above equation can never have three rational solutions.

(Josef Greilhuber) □