52^{nd} Austrian Mathematical Olympiad

National Competition—Final Round—Solutions

4th/5th June 2021

Problem 1. *Let a, b, c be pairwise distinct natural numbers.*

Prove that

$$\frac{a^3+b^3+c^3}{3} \geq abc + a + b + c.$$

When does equality hold?

(Karl Czakler)

Solution. It is well-known and easily verified that

$$a^3+b^3+c^3-3abc = \frac{1}{2}(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2). \tag{1}$$

Assume without loss of generality that $a > b > c \geq 0$. Since the numbers are integers, we obtain $a-b \geq 1$, $b-c \geq 1$ and $a-c \geq 2$.

Equation (1) now implies

$$a^3+b^3+c^3-3abc \geq \frac{1}{2}(a+b+c)(1+1+4) = 3(a+b+c)$$

as desired.

Equality holds for $a = b+1$, $b = c+1$ and $a = c+2$, which are exactly the triples $(t+2, t+1, t)$ where $t \geq 0$ is an integer, and for all their permutations.

(Karl Czakler) □

Problem 2. *Mr. Precise wants to take his tea cup out of the microwave precisely at the front. The microwave of Mr. Precise is not precisely cooperative.*

More precisely, the two of them play the following game:

Let n be a positive integer. The rotating plate of the microwave takes n seconds for a full turn. Each time the microwave is turned on, the plate is turned clockwise or counterclockwise for an integer number of seconds such that the tea cup can end up in n possible positions. One of these positions is marked „front“.

At the start of the game, the microwave rotates the tea cup in one of these positions. Afterwards, for each move, Mr. Precise enters the integer number of seconds and the microwave decides whether to turn clockwise or counterclockwise.

For which n can Mr. Precise ensure that after a finite number of moves, he can take out the tea cup of the microwave precisely from the front position?

(Birgit Vera Schmidt)

Answer. Mr. Precise can ensure his victory when n is a power of 2.

Solution. We label the positions consecutively $0, 1, \ldots, n-1$ where 0 is the front position.

If n is a power of 2, say $n = 2^k$, Mr. Precise can simply always put in the current position as number of seconds. If the microwave turns the plate backwards, the tea cup will end up front immediately. Otherwise, the number of the position will be doubled and reduced modulo 2^k at each turn and therefore be divisible by 2^k after at most k turns. This means that the tea cup ends up front.

Now, let $n = 2^k \cdot m$, where $m > 1$ is an odd number.

We will show that the microwave can always choose a number not divisible by m.

This is clearly true for the first position, for example by choosing the position 1. After that, the tea cup is in a certain position p not divisible by m and Mr. Precise puts in s seconds. If both $p+s$ and $p-s$ were divisible by m, this would also be true for the sum, so that $m \mid 2p$. Since m is odd, this implies $m \mid p$ which is wrong.

(Stephan Pfannerer) □

Problem 3. *Determine all triples (a, b, c) of integers $a \geq 0$, $b \geq 0$ und $c \geq 0$ that satisfy the equation*

$$a^{b+20}(c-1) = c^{b+21} - 1.$$

(Walther Janous)

Answer. $\{(1, b, 0) : b \in \mathbb{Z}_{>0}\} \cup \{(a, b, 1) : a, b \in \mathbb{Z}_{>0}\}$

Solution. One can first see that the right side factors:

$$a^{b+20}(c-1) = (c^{b+20} + c^{b+19} + \cdots + c + 1)(c-1).$$

The case $c = 1$ will be handled separately (and is very simple). For $c \neq 1$ the equation simplifies to

$$a^{b+20} = c^{b+20} + c^{b+19} + \cdots + c + 1.$$

We therefore distinguish the two cases for c.

- $c = 1$ leads to
$$0 = 0.$$
Therefore, in this case, arbitrary natural numbers a and b are solutions.

- For $c \neq 1$ we can divide by $c - 1$ (see the above equations) and get the equivalent equation
$$a^{b+20} = c^{b+20} + c^{b+19} + \cdots + c + 1.$$
Obviously,
$$c^{b+20} + c^{b+19} + \cdots + c + 1 > c^{b+20}.$$
Therefore $a \geq c + 1$ must hold. Because of the binomial theorem we, thus, obtain
$$\begin{aligned} a^{b+20} &\geq (c+1)^{b+20} \\ &= c^{b+20} + \binom{b+20}{1} c^{b+19} + \cdots + \binom{b+20}{b+19} c + 1 \\ &\geq c^{b+20} + c^{b+19} + \cdots + c + 1 \\ &= a^{b+20}. \end{aligned}$$
Hence, both inequalities must be equations.
We consider the second inequality in particular. Because of
$$\binom{b+20}{1} = b + 20 > 1$$
this can only be an equation if $c = 0$. In the case of $c > 0$, the second inequality is strict and therefore leads to a contradiction and there is no solution.
In the remaining case $c = 0$, the resulting equation
$$a^{b+20} = 1$$
is easy to solve. Since a is a natural number, $a = 1$. (This also follows from the necessary relationship $a = c + 1$.) Hence, in this case b may be any natural number.

Alternatively, one can see in the case $c > 0$ that

$$\begin{aligned} c^{b+20} &< c^{b+20} + c^{b+19} + \cdots + c + 1 \\ &< c^{b+20} + \binom{b+20}{1} c^{b+19} + \cdots + \binom{b+20}{b+19} c + 1 \\ &= (c+1)^{b+20}. \end{aligned}$$

So a^{b+20} is in this case strictly between c^{b+20} and $(c+1)^{b+20}$, which is impossible for natural numbers. (The case $c = 0$ must then be treated separately as above.)

(Michael Drmota) $\square$

Problem 4. *Let α be a real number.*

Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x) + y) = f(x^2 - y) + \alpha f(x) y$$

for all $x, y \in \mathbb{R}$.

(Walther Janous)

Solution. First, we set $y = (x^2 - f(x))/2$ which gives $\alpha f(x)(x^2 - f(x))/2 = 0$. Now, we distinguish the cases $\alpha \neq 0$ und $\alpha = 0$.

(a) In the case $\alpha \neq 0$, this equation implies $f(x) = 0$ or $f(x) = x^2$ for each x separately.

In particular, $f(0) = 0$.

- It is easily verified that $f(x) = 0$, $x \in \mathbb{R}$, is a solution.
- Next, we investigate the function $f(x) = x^2$, $x \in \mathbb{R}$. The functional equation becomes

$$(x^2 + y)^2 = (x^2 - y)^2 + \alpha x^2 y \iff 2x^2 y = -2x^2 y + \alpha x^2 y \iff (\alpha - 4)x^2 y = 0$$

which holds for all x and y exactly when $\alpha = 4$. Therefore, for $\alpha = 4$, there is an additional solution $f(x) = x^2$, $x \in \mathbb{R}$.
- It remains to investigate the case, where there are numbers x, $y \in \mathbb{R} \setminus \{0\}$ with $f(y) = y^2$ and $f(x) = 0$. Suppose that x and y were two such numbers.
Then the original functional equation becomes $f(y) = f(x^2 - y)$. Because of $f(y) = y^2 \neq 0$, we have $f(x^2 - y) \neq 0$ and therefore $f(x^2 - y) = (x^2 - y)^2$. This implies

$$y^2 = (x^2 - y)^2 = x^4 - 2x^2 y + y^2,$$

i.e.. $y = x^2/2$, so that y is the only number with $f(y) = y^2$ and $f(z) = 0$ for all $z \in \mathbb{R} \setminus \{y\}$. Repeating this argument, we obtain $y = z^2/2$ for all $z \in \mathbb{R} \setminus \{y\}$, a contradiction.

(b) For $\alpha = 0$, the functional equation becomes

$$f(f(x) + y) = f(x^2 - y). \tag{2}$$

It is easy to check that constant functions and the function $f(x) = -x^2$ are solutions.

Now, assume that there is a real number a with $f(a) = b \neq -a^2$. We define $d = b + a^2 \neq 0$.

Putting $x = a$ in the functional equation (2) gives $f(b + y) = f(a^2 - y)$ for all $y \in \mathbb{R}$. With $y = z - b$, we obtain $f(z) = f(d - z)$ for all $z \in \mathbb{R}$. Using $x = z$ and $x = d - z$ in the functional equation (2), we get

$$f(z^2 - y) = f(f(z) + y) = f(f(d - y) + y) = f((d - z)^2 - y).$$

Therefore,

$$f(z^2 - y) = f((d - z)^2 - y)$$

for all real numbers y and z. With $y = z^2$, we obtain $f(0) = f(d^2 - 2dz)$ for all $z \in \mathbb{R}$. Because of $d \neq 0$ the second argument attains all real numbers, so that f is constant. This proves that there are no other solutions.

(Walther Janous) □

Problem 5. *Let $ABCD$ be an inscribed convex quadrilateral with diagonals AC and BD. Each of the four vertices is reflected on the diagonal it does not lie on.*

Prove that the resulting four points lie on a common circle or a common line.

(a) Investigate when the four resulting points lie on a common line and give a simple equivalent condition for the quadrilateral $ABCD$.

(b) Prove that in all other cases, the four resulting points lie on a common circle.

(Theresia Eisenkölbl)

Solution. (a) We denote the reflections of A, B, C and D with A', B', C' resp. D' and we denote the intersection of the diagonals with S. Since the points A and C are reflected in the same line BD and the point S remains invariant under this reflection, the whole line ASC becomes $A'SC'$ after reflection in BD. Analogously, the line BSD becomes $B'SD'$ after reflection in AC.

If we denote the smaller angle between the two diagonals by φ, these two actions on the lines correspond to a rotation of the line AC with center S in direction BD with rotation angle 2φ and a rotation of the line BD with center S in direction AC with rotation angle 2φ.

Therefore, the angle between the lines $A'SC'$ and $B'SD'$ is the angle 3φ. This has to be a multiple of 180°, so that the original angle has to be 0° or 60°. The first case is not possible since the points of the inscribed quadrilateral cannot lie on a line.

We obtain that the four new points lie on a line if and only if the diagonals of the given inscribed quadrilateral make an angle of 60°.

(b) Since the reflections do not only preserve the collinearity of ASC and BSD, but also the position of S between the two points and the distances to the two points, we want to use the power of S with respect to the circle $ABCD$.

Because of the reflections, we have

$$SA = SA', \; SB = SB', \; SC = SC', \; SD = SD',$$

and since $ABCD$ is an inscribed quadrilateral, we have

$$SA \cdot SC = SB \cdot SD.$$

Therefore, we obtain

$$SA' \cdot SC' = SA \cdot SC = SB \cdot SD = SB' \cdot SD'.$$

Since the two lines $A'SC'$ and $B'SD'$ do not coincide in this case, we can apply the properties of the power of a point in reverse, and we get that A', B', C' and D' lie on a circle.

(Theresia Eisenkölbl) □

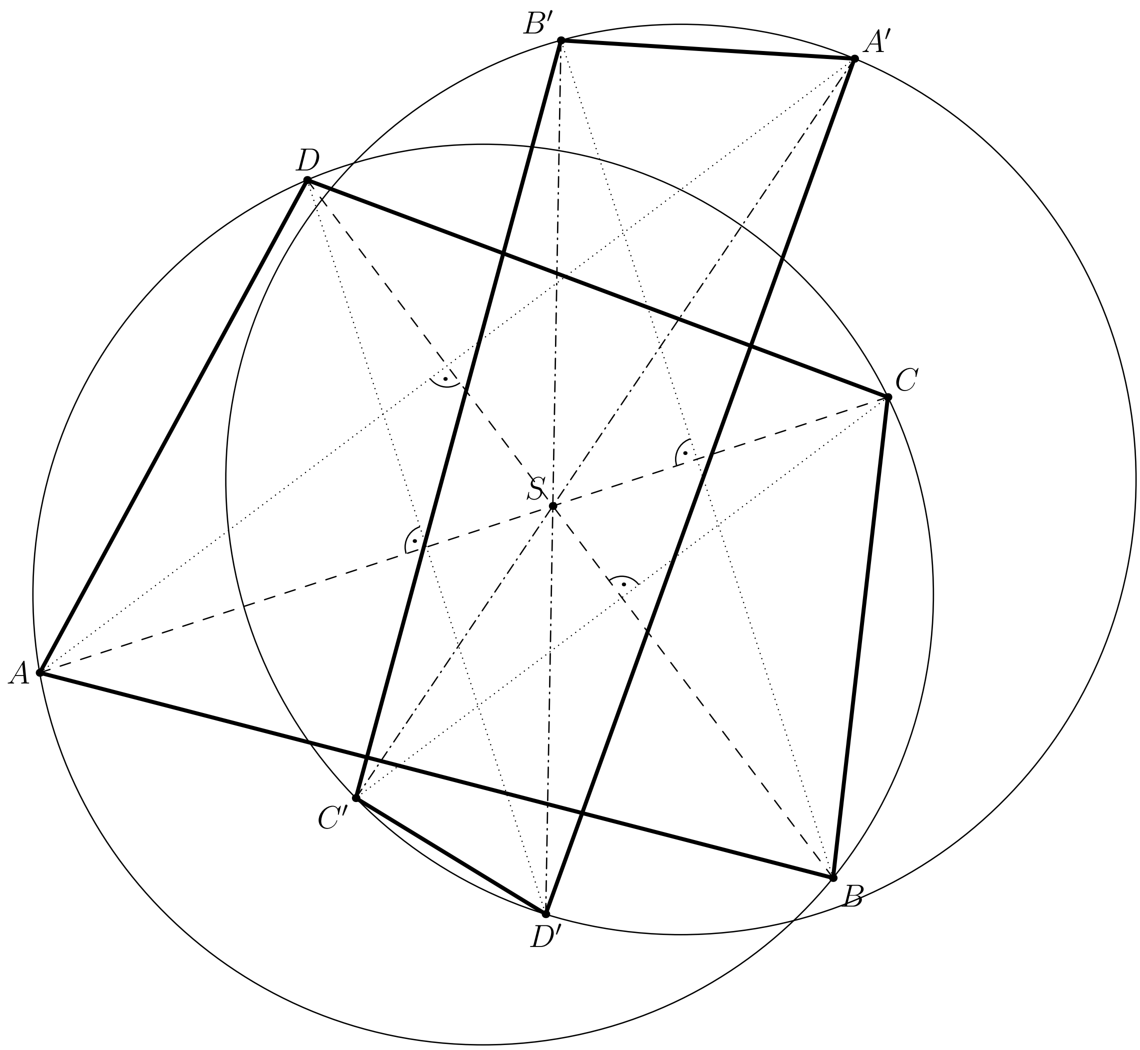

Figure 1: Problem 5

Problem 6. *Suppose that p is an odd prime number and M a set of $\frac{p^2+1}{2}$ integer squares.*

Investigate if one can choose p elements of this set so that the arithmetic mean of these p elements is an integer.

(Walther Janous)

Answer. Yes.

Solution. The idea is to choose from the $\frac{p^2+1}{2}$ square numbers p numbers that are in the same residue class modulo p. Obviously, the sum of these p numbers is then divisible by p and thus the arithmetic mean is an integer.

It is known that the square numbers do not run through all residue classes modulo p, but only through $1 + \frac{p-1}{2} = \frac{p+1}{2}$ ones. (On the one hand, this is the residue class 0 if one squares a number divisible by p. Because of $a^2 \equiv (p-a)^2 \bmod p$, the squares of numbers a that are not divided by p run through a maximum of half of the $p-1$ nonzero residue classes. On the other hand, $x^2 \equiv y^2 \bmod p$ gives the relation $p \mid (x-y)(x+y)$ and so $x \equiv y \bmod p$ or $x \equiv -y \bmod p$. Therefore, the squares of numbers a, which are not divisible by p, run through exactly half of the $p-1$ residue classes different from zero.)

We now divide the $\frac{p^2+1}{2}$ square numbers into the $\frac{p+1}{2}$ residue classes that correspond to square numbers. Because of the pigeon hole principle, there is therefore a residue class, that contains at least

$$\left\lceil \frac{(p^2+1)/2}{(p+1)/2} \right\rceil$$

numbers.

Because of

$$\frac{(p^2+1)/2}{(p+1)/2} = \frac{p^2+1}{p+1} = \frac{p^2+p}{p+1} - \frac{p-1}{p+1} = p - \frac{p-1}{p+1}$$

and $0 < \frac{p-1}{p+1} < 1$ it follows that

$$\left\lceil \frac{(p^2+1)/2}{(p+1)/2} \right\rceil = p,$$

what was to be shown.

(Michael Drmota) □