# $49^{\text{th}}$ Austrian Mathematical Olympiad

National Competition (Final Round, part 2)—Solutions

531/1st June 2018

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**Problem 1.** *Let $\alpha \neq 0$ be a real number.*

*Find all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with*

$$f(f(x)+y) = \alpha x + \frac{1}{f\left(\frac{1}{y}\right)}$$

*for all $x$, $y \in \mathbb{R}_{>0}$.*

(Walther Janous)

*Solution.* Answer: If $\alpha = 1$, the only solution is $f(x) = x$. For other values of $\alpha$, there is no solution.

We must have $\alpha > 0$, otherwise, the right-hand side becomes negative for large values of $x$. By using $x$ in the given equation, we can immediately conclude that $f$ is an injective function. Furthermore, since we can choose arbitrary values for $x$ on the right-hand side, we conclude that $f$ is surjective on an interval $(a, \infty)$. By choosing small values of $x$ and large values of the function on the right-hand side, we conclude that the right-hand side takes all positive values, so the function $f$ is surjective.

Now, we replace $y$ with $f(y)$ and obtain

$$f(f(x)+f(y)) = \alpha x + \frac{1}{f\left(\frac{1}{f(y)}\right)}. \tag{1}$$

The left-hand side is symmetric in $x$ and $y$, therefore

$$\alpha x + \frac{1}{f\left(\frac{1}{f(y)}\right)} = \alpha y + \frac{1}{f\left(\frac{1}{f(x)}\right)}.$$

If we choose an arbitrary fixed value for $y$, we get

$$\frac{1}{f\left(\frac{1}{f(x)}\right)} = \alpha x + C.$$

We substitute this identity into Equation (1) and get

$$f(f(x)+f(y)) = \alpha x + \alpha y + C.$$

Because of injectivity, this implies

$$f(x)+f(y) = f(z)+f(w), \text{ if } x+y = z+w. \tag{2}$$

In particular, we have

$$f(x+1)+f(y+1) = f(x+y+1)+f(1) \qquad \text{for } x,\ y \geq 0.$$

With $g(x) = f(x+1)$, we get for the function $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that

$$g(x)+g(y) = g(x+y)+g(0).$$

We put $h(x) = g(x) - g(0)$ and get $h(x) \geq -g(0)$ and the Cauchy functional equation

$$h(x)+h(y) = h(x+y).$$

If we had $h(t) < 0$ for some $t > 0$, then the values $h(nt) = nh(t)$ would be arbitrarily small for large positive integers $n$. But this is impossible because of the lower bound, so we have $h(x) \geq 0$ for all $x \geq 0$ and we get for $0 < u < v$ that $h(v) = h(u) + h(v-u) \geq h(u)$.

Therefore, the function $h(x)$ is a monotone solution of the Cauchy functional equation which has to be of the form $h(x) = cx$. This implies that for $x > 1$ we also have $f(x) = h(x-1) + g(0) = cx + d$ for some constants $c$ and $d$. But this also true for $0 < x \leq 1$ which can be seen by plugging $y = 3$, $z = 2$ and $w = x + 1$ into Equation (2).

Since $f$ is surjective, the contant term has to be 0 (otherwise, small positive values could not be reached by $f$ or negative values would be reached). Putting $f(x) = cx$ into the original equation and equating coefficients gives the conditions $c^2 = \alpha$ and $c^2 = 1$. Since $c$ has to be positive, we obtain the only solution $f(x) = x$ if $\alpha = 1$.

*(Theresia Eisenkölbl)* □

**Problem 2.** *Let $A$, $B$, $C$ and $D$ be four different points lying on a common circle in this order. Assume that the line segment $AB$ is the (only) longest side of the inscribed quadrilateral $ABCD$.*

*Prove that the inequality*

$$AB + BD > AC + CD$$

*holds.*

(Karl Czakler)

*Solution.* Let $S$ denote the common point of the diagonals, and let $a = AB$ and $c = CD$.

Since $ABCD$ is an inscribed quadrilateral, triangles $ABS$ and $DCS$ are similar. It follows that numbers $r$ and $s$ must exist, such that $AS = sa$, $BS = ra$, $DS = sc$ and $CS = rc$ hold. The inequality under consideration can therefore be written in the form

$$a + ra + sc > sa + rc + c.$$

This is equivalent to

$$a(1 + r - s) > c(1 + r - s),$$

which is certainly correct, since $a > c$ is given and the triangle inequality in $ABS$ implies $1 + r > s$.

*(Theresia Eisenkölbl)* □

**Problem 3.** *There are $n$ children in a room. Each child has at least one piece of candy. In Round 1, Round 2, etc., additional pieces of candy are distributed among the children according to the following rule:*

*In Round $k$, each child whose number of pieces of candy is relatively prime to $k$ receives an additional piece.*

*Show that after a sufficient number of rounds the children in the room have at most two different numbers of pieces of candy.*

(Theresia Eisenkölbl)

*Solution.* We observe that a child that has $k - 1$ or $k + 1$ pieces of candy at the start of Round $k$ will receive an additional piece because of $\gcd(k, k \pm 1) = 1$ and be in the same situation in the next round. Furthermore, in each round, the number of the round will increase by 1 and the number of pieces of each child will increase by 0 or by 1. Therefore, for each child, the difference of pieces and the round number is positive or zero at the start and after each round will either remain equal or drop by 1. Since we have already seen that the difference is stable at $-1$, it cannot drop below $-1$.

Therefore, it remains to show that the difference $+1$ and $-1$ are the only ones that can stay constant forever which will prove that for each child the number of pieces of candy will eventually drop to $k - 1$ or $k + 1$.

If the difference is 0, the child has $k$ pieces of candy. For $k = 1$, the child receives a piece, but receives nothing in the following round, so the difference drops to $-1$ after two steps. For $k > 1$, the child immediately receives nothing and the difference drops to $-1$.

If the difference $d$ is bigger than 1, then there must occur a round with a number divisible by $d$ after at most $d$ steps. Either the difference already drops before this round or this $d$ will be a common divisor of round number and candy piece number, therefore the difference will drop by 1 after at most $d$ steps.

This proves that after sufficiently long time all children will have $k-1$ or $k+1$ pieces of candy at the start of Round $k$ and all of them will receive one additional piece during each round forever after.

*(Theresia Eisenkölbl)* □

**Problem 4.** *Let $ABC$ be a triangle and $P$ a point inside the triangle such that the centers $M_B$ and $M_A$ of the circumcircles $k_B$ and $k_A$ of triangles $ACP$ and $BCP$, respectively, lie outside the triangle $ABC$. In addition, we assume that the three points $A$, $P$ and $M_A$ are collinear as well as the three points $B$, $P$ and $M_B$. The line through $P$ parallel to side $AB$ intersects circles $k_A$ and $k_B$ in points $D$ and $E$, respectively, where $D$, $E \neq P$.*

*Show that $DE = AC + BC$.*

(Walther Janous)

*Solution.* We put $\varphi := \angle CBP$, cf. Figure 1. Then we get for the corresponding central angle $\angle CM_AP =$

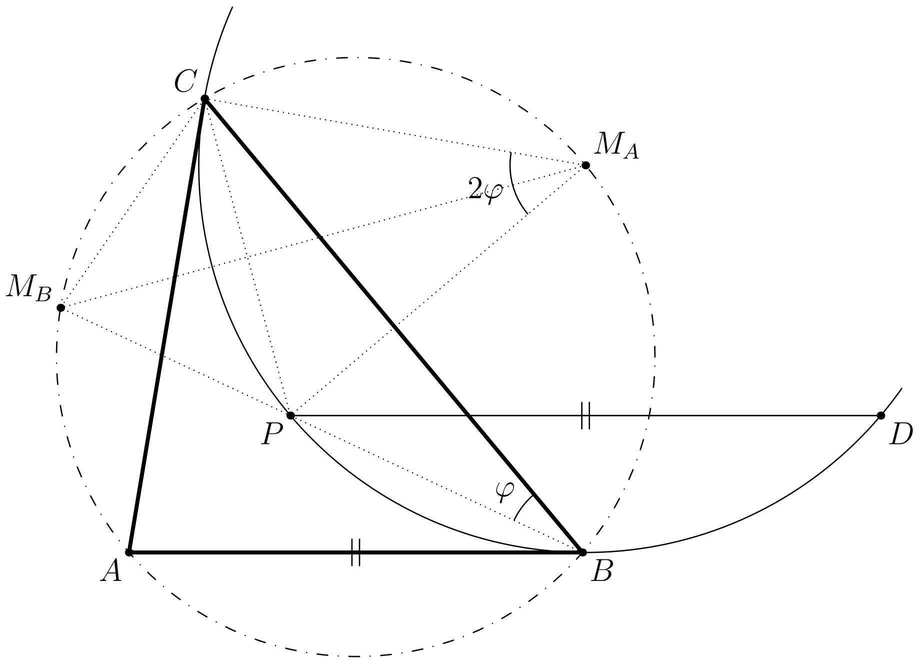


Figure 1: Problem 4

$2\varphi$. Since $M_ACM_BP$ is a deltoid having $M_AM_B$ as its axis of symmetry, we deduce $\angle CM_AM_B = \varphi = \angle CBM_B$. Therefore, $B$ and analogously $A$ lie on the circumcircle of $M_ACM_B$. In other words, the two centers $M_A$ and $M_B$ lie on the circumcircle of $ABC$.

Thus $M_A$ and $M_B$ are the south poles corresponding to vertices $A$ and $B$, respectively. As a result, $P$ is the incenter of triangle $ABC$. Hence $\angle PBA = \angle CBP$ and because of $PD \parallel AB$ also $\angle CBP = \angle BPD$ holds true. This means that $PBDC$ is an isosceles trapezoid with diagonals of equal lengths and $PD = BC$ follows.

In a similar way $PE = AC$ can be shown. Finally, by addition we arrive at the claim $DE = AC+BC$.

*(Clemens Heuberger, Walther Janous)* □

**Problem 5.** *On a circle 2018 points are marked.*

*Each of these points is labeled with an integer. Let each number be larger than the sum of the preceding two numbers in clockwise order.*

*Determine the maximal number of positive integers that can occur in such a configuration of 2018 integers.*

(Walther Janous)

*Solution.* Let the points be labeled $a_0$, $a_1$, ..., $a_{2017}$ clockwise with cyclical notation, i.e., $a_{k+2018} = a_k$ for all integers $k$.

> *Lemma.* In a valid configuration, no two neighbouring numbers can be both non-negative.

> *Proof.* Assume that there exist neighbouring numbers $a_{k-1}$ and $a_k$ which are both non-negative. We get $a_{k+1} > a_k + a_{k-1} \geq a_k$, with the first inequality following from the problem statement and the second from $a_{k-1} \geq 0$. Since now also $a_k$ and $a_{k+1}$ are both non-negative, we analogously get $a_{k+2} > a_{k+1}$, then $a_{k+3} > a_{k+2}$, and so on, until we have $a_{k+2018} > a_{k+2017} > \cdots > a_{k+1} > a_k = a_{k+2018}$, a contradiction. ∎

Therefore at most every second number can be non-negative. Next we will show that these are still too many non-negative numbers.

> *Lemma.* In a valid configuration, it is not possible that every second number is non-negative.

> *Proof.* Assume that this is the case, so w.l.o.g. $a_{2k} \geq 0$ and $a_{2k+1} < 0$ for all integers $k$. Then we get $a_3 > a_2 + a_1 \geq a_1$, where again the first inequality follows from the problem statement and the second from $a_2 \geq 0$. Analogously we get $a_5 > a_3$, then $a_7 > a_5$, et cetera, until we have $a_1 = a_{2019} > a_{2017} > a_{2015} > \cdots > a_3 > a_1$, a contradiction. ∎

We can therefore summarize: A configuration with more than 1009 non-negative numbers is not possible because otherwise by the pigeonhole principle we would have two neighbouring non-negative numbers, which is not allowed according to the first lemma. A configuration with exactly 1009 non-negative numbers contradicts either the first or the second lemma.

With 1008 positive and 1010 negative numbers we find for example the configuration

$$-4035, 1, -4033, 1, -4031, 1, -4029, \ldots, 1, -2021, 1, -2019, -2017,$$

which we can easily check for correctness.

*(Birgit Vera Schmidt)* □

**Problem 6.** *Determine all digits $z$ such that for each integer $k \geq 1$ there exists an integer $n \geq 1$ with the property that the decimal representation of $n^9$ ends with at least $k$ digits $z$.*

(Walther Janous)

*Solution.* Answer: This is possible for $z \in \{0, 1, 3, 7, 9\}$.

For $z = 0$ we easily find $10^l$ with any sufficiently large integer $l$ such that $9l \geq k$.

For $z \in \{2, 4, 6, 8\}$ the number $n^9$ is even and therefore also $n$ must be even, and hence $n^9$ must be divisible by $2^9$. However, numbers ending with 222, 444 or 666 are already not divisible by 8, and numbers ending with 8888 are not divisible by 16. Therefore, there does not exist a solution for these values of $z$.

Similarly, for $z = 5$ the number $n^9$ is divisible by 5, therefore also $n$ itself is divisible by 5 and therefore, $n^9$ must even be divisible by $5^9$. However, numbers ending with 55 are not divisible by 25.

For $z \in \{1,3,7,9\}$, let $b := (\underbrace{zzz\dots z}_{k})_{10}$. Since $\gcd(9,\varphi(10^k)) = \gcd(9, 4\cdot 10^{k-1}) = 1$, by the Euclidean algorithm there exist numbers $x$ and $y$ such that $9x + \varphi(10^k)y = 1$. We claim that $n := b^x$ has the desired property. Because of $\gcd(b, 10^k) = 1$ this can be easily demonstrated using Euler-Fermat:

$$(b^x)^9 = b^{9x} \equiv b^{9x+\varphi(10^k)y} = b^1 = b \pmod{10^k}.$$

*(Clemens Heuberger, Walther Janous)* □