47^{th} Austrian Mathematical Olympiad

National Competition (Final Round, part 2) – Solutions

May 25, 2016 and May 26, 2016

Problem 1. *Let $\alpha \in \mathbb{Q}^+$. Determine all functions $f\colon \mathbb{Q}^+ \to \mathbb{Q}^+$ such that*

$$f\Big(\frac{x}{y} + y\Big) = \frac{f(x)}{f(y)} + f(y) + \alpha x$$

holds for all x, $y \in \mathbb{Q}^+$.

Here, $\mathbb{Q}^+$ denotes the set of positive rational numbers.

(Walther Janous)

Solution. Setting $y = x$ and $y = 1$ yields

$$f(x+1) = 1 + f(x) + \alpha x \tag{1}$$

and

$$f(x+1) = \frac{f(x)}{f(1)} + f(1) + \alpha x, \tag{2}$$

respectively. Equating (1) and (2) implies

$$f(x)\Big(1 - \frac{1}{f(1)}\Big) = f(1) - 1.$$

As f cannot be constant due to (1), we obtain $f(1) = 1$. By induction, we get

$$f(x) = \frac{\alpha}{2}x(x-1) + x \qquad \text{for all } x \in \mathbb{Z}^+. \tag{3}$$

In particular, this implies $f(2) = \alpha + 2$ and $f(4) = 6\alpha + 4$. Setting $x = 4$ and $y = 2$ in the functional equation yields

$$\alpha^2 - 2\alpha = 0.$$

Thus we must have $\alpha = 2$ in order to obtain solutions. From now on, we only consider this case.

From (3), we obtain $f(x) = x^2$ for $x \in \mathbb{Z}^+$. By induction, we obtain that for $x \in \mathbb{Q}^+$ and $n \in \mathbb{Z}^+$, from the relation $f(x+n) = (x+n)^2$ it follows that $f(x) = x^2$.

Let now $\frac{a}{b} \in \mathbb{Q}^+$ with a, $b \in \mathbb{Z}^+$. We set $x = a$ and $y = b$ and obtain

$$f\Big(\frac{a}{b} + b\Big) = \frac{a^2}{b^2} + b^2 + 2a = \Big(\frac{a}{b} + b\Big)^2.$$

The above remark implies that $f(\frac{a}{b}) = \big(\frac{a}{b}\big)^2$. It is easily verified that $f(x) = x^2$ is indeed a solution.

Thus there is no solution for $\alpha \neq 2$ and the solution $f(x) = x^2$ for $\alpha = 2$.

(Theresia Eisenkölbl) □

Problem 2. *Let ABC be a triangle. Its incircle meets the sides BC, CA and AB in the points D, E and F, respectively. Let P denote the intersection point of ED and the line perpendicular to EF and passing through F, and similarly let Q denote the intersection point of EF and the line perpendicular to ED and passing through D.*

Prove that B is the mid-point of the segment PQ.

(Karl Czakler)

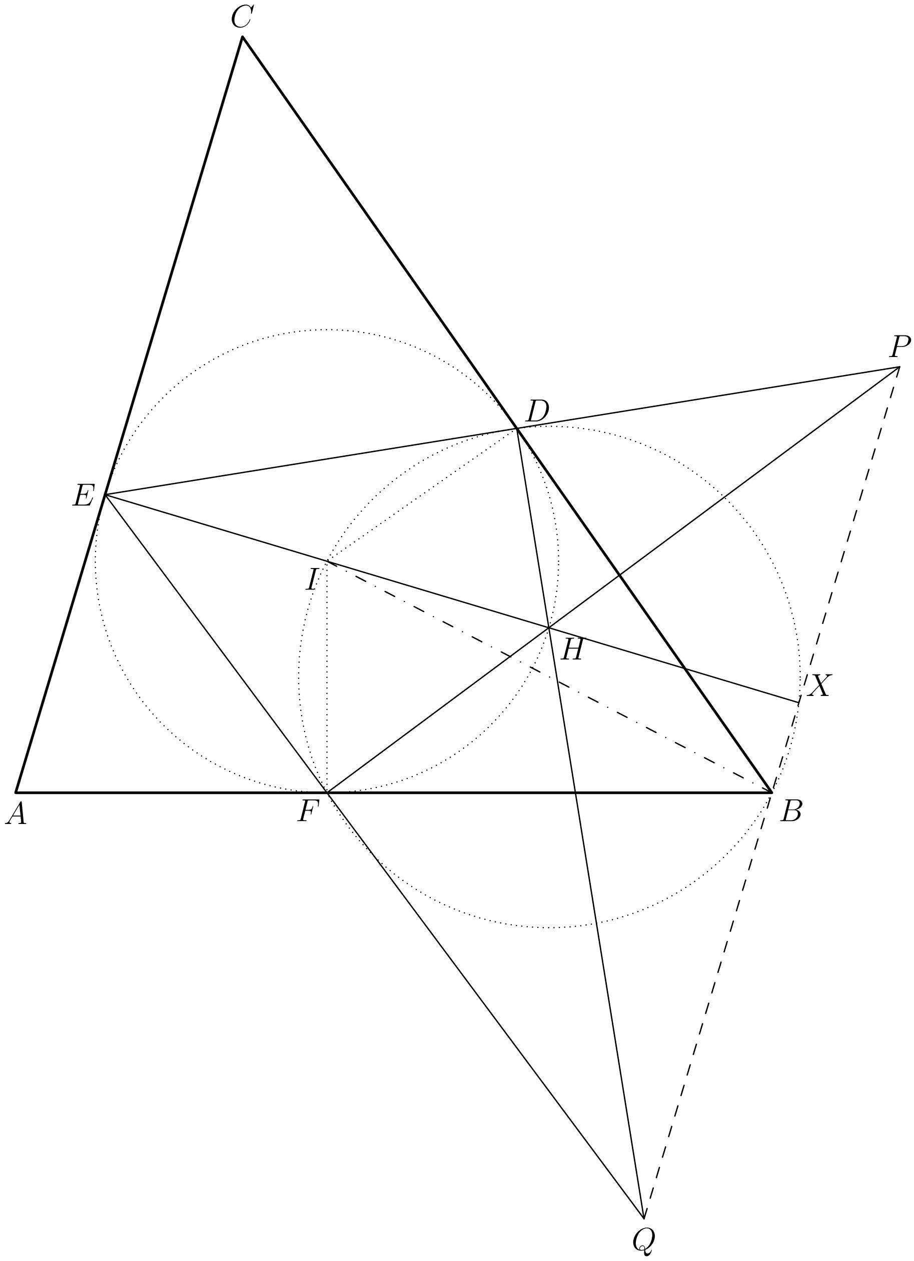

Figure 1: Problem 2

Solution. Let H be the common point of PF and QD, as can be seen in Figure 1. Since $\angle EDH$ and $\angle HFE$ are both right angles, HE is a diameter of the incircle of ABC. Now let X denote the common point of EH and PQ. We see that H is the orthocenter of the triangle EPQ, and X, D and F are the feet of the altitudes in this triangle. The incenter I of ABC is also the mid-point of an altitude segment. It follows that points I, F, X and D all lie on the nine-point circle of EPQ.

Because of the right angles in F and D, we know that I, F, D and B lie on a common circle. This circle is the nine-point circle of EPQ. For the same reason, B is the diametrically opposed point to I on the nine-point circle of EPQ.

It is well known that the mid-point of each altitude segment lies diametrically opposed to the mid-point of the corresponding side of the triangle. (Note the right angle in X.) We therefore see that B must be the midpoint of PQ, as we had set out to show.

(Sara Kropf) □

Problem 3. *Consider arrangements of the numbers* 1 *through* 64 *on the squares of an* 8×8 *chess board, where each square contains exactly one number and each number appears exactly once.*

A number in such an arrangement is called super-plus-good*, if it is the largest number in its row and at the same time the smallest number in its column.*

Prove or disprove each of the following statements:

(a) Each such arrangement contains at least one super-plus-good number.

(b) Each such arrangement contains at most one super-plus-good number.

(Gerhard J. Woeginger)

Solution. (a) This is wrong. For example, one might place the numbers from 1 to 8 along the main diagonal and the numbers from 57 to 64 along the secondary diagonal:

1	9	10	11	12	13	14	**57**
15	**2**	16	17	18	19	**58**	20
21	22	**3**	23	24	**59**	25	26
27	28	29	**4**	**60**	30	31	32
33	34	35	**61**	**5**	36	37	38
39	40	**62**	41	42	**6**	43	44
45	**63**	46	47	48	49	**7**	50
64	51	52	53	54	55	56	**8**.

Therefore the numbers from 1 to 8 are column minima, whereas the numbers from 57 to 64 are row maxima. Therefore, no number is at the same time column minimum and row maximum, so no number is super-plus-good.

(b) This is true. Denote the number in the ath row and bth column by $F(a, b)$. Assume that there exist two super-plus-good numbers, and let (i, j) and (r, s) be the coordinates of these two numbers. Since all numbers are different, the row maxima and column minima are unique. Therefore no row and no column may contain more than one super-plus-good number, so $i \neq r$ and $j \neq s$ must hold. Then

$$\begin{aligned}
F(i,j) &> F(i,s) && \text{(because } F(i,j) \text{ is row maximum),}\\
F(i,j) &< F(r,j) && \text{(because } F(i,j) \text{ is column minimum),}\\
F(r,s) &> F(r,j) && \text{(because } F(r,s) \text{ is row maximum),}\\
F(r,s) &< F(i,s) && \text{(because } F(r,s) \text{ is column minimum).}
\end{aligned}$$

These four inequalities lead to the following contradiction:

$$F(i,j) > F(i,s) > F(r,s) > F(r,j) > F(i,j).$$

(Gerhard J. Woeginger) □

Problem 4. *Let a, b, $c \geq -1$ be real numbers with $a^3 + b^3 + c^3 = 1$. Prove that*

$$a + b + c + a^2 + b^2 + c^2 \leq 4.$$

When does equality hold?

(Karl Czakler)

Solution. We note that

$$1 - x - x^2 + x^3 = (1-x)^2(1+x) \geq 0 \tag{4}$$

holds for all real numbers $x \geq -1$. Here, equality holds iff $x = \pm 1$.

Thus we have $x + x^2 \leq 1 + x^3$ for $x = a$, b, c and therefore

$$a + a^2 + b + b^2 + c + c^2 \leq 1 + a^3 + 1 + b^3 + 1 + c^3 = 3 + 1 = 4.$$

As equality in (4) holds for ± 1 and the sum of the cubes equals 1, equality in the original inequality holds iff (x, y, z) is a permutation of $(1, 1, -1)$.

(Theresia Eisenkölbl) □

Problem 5. *Consider a board consisting of $n \times n$ unit squares where $n \geq 2$. Two cells are called neighbors if they share a horizontal or vertical border. In the beginning, all cells together contain k tokens. Each cell may contain one or several tokens or none.*

In each turn, choose one of the cells that contains at least one token for each of its neighbors and move one of those to each of its neighbors. The game ends if no such cell exists.

(a) Find the minimal k such that the game does not end for any starting configuration and choice of cells during the game.

(b) Find the maximal k such that the game ends for any starting configuration and choice of cells during the game.

(Theresia Eisenkölbl)

Solution. 1. If each cells contains one token less than the number of its neighbors, the game cannot even start. On the other hand, if there is one token more then by the pigeon-hole principle there will always exist at least one cell with sufficient tokens to make the next move.

Therefore, the desired quantity is the sum of all numbers of neighbors minus the number of all cells plus 1. If one adds $4n$ cells around the n^2 given cells, each original cell has four neighbors and each new cell has contributed one neighbor.

We get $k = (4n^2 - 4n) - n^2 + 1 = 3n^2 - 4n + 1$.

2. It is easy to see that an unlimited number of turns must eventually have brought tokens to all cells and that also every pair of neighbors must have exchanged tokens because otherwise tokens would accumulate in unlimited number in one of the inactive cells.

   But each time, a neighboring pair first exchanges tokens, we can reserve this first token to stay always between these two neighbors. Therefore, the game will certainly end if there are less tokens then neighboring pairs.

   Conversely, if the number of tokens equals the number of neighboring pairs, we can find a never-ending game in the following way: Color the cells black and white in a checkerboard fashion and assign to each black cell a number of tokens that equals the number of its neighbors. Now we will simply choose all the black cells until all tokens are on the white cells, then repeat with the white cells and then iterate from the start.

   The desired quantity is therefore the number of neighboring pairs minus 1. Since the number of neighboring pairs is half of the first expression in the computation of part 1, we get $k = 2n^2 - 2n - 1$.

   (Theresia Eisenkölbl) □

Problem 6. *Let a, b, c be integers such that*

$$\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a}$$

is an integer.

Prove that each of the numbers

$$\frac{ab}{c}, \ \frac{ac}{b} \quad \text{and} \quad \frac{bc}{a}$$

is an integer.

(Gerhard J. Woeginger)

Solution. Set $u := ab/c$, $v := ac/b$ and $w := bc/a$. By assumption, $u + v + w$ is an integer. It is easily seen that $uv + uw + vw = a^2 + b^2 + c^2$ and $uvw = abc$ are integers, too.

According to Vieta's formulæ, the rational numbers u, v, w are the roots of a cubic polynomial $x^3 + px^2 + qx + r$ with integer coefficients. As the leading coefficient is 1, these roots are integers.

(Gerhard J. Woeginger) □