54$^{\text{th}}$ Austrian Mathematical Olympiad

National Competition—Preliminary Round

29th April 2023

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1. Let a, b, c, d be real numbers with $0 < a, b, c, d < 1$ and $a + b + c + d = 2$. Show that
$$\sqrt{(1-a)(1-b)(1-c)(1-d)} \leq \frac{ac+bd}{2}.$$
Are there infinitely many cases of equality?

(Josef Greilhuber)

2. Let ABC be a triangle. Let P be the point on the extension of BC beyond B such that $BP = BA$. Let Q be the point on the extension of BC beyond C such that $CQ = CA$.

   Prove that the circumcenter O of the triangle APQ lies on the angle bisector of the angle $\angle BAC$.

(Karl Czakler)

3. Let n be a positive integer. What proportion of the non-empty subsets of $\{1, 2, \ldots, 2n\}$ has a smallest element that is odd?

(Birgit Vera Schmidt)

4. Determine all pairs of positive integers (n, k) for which
$$n! + n = n^k$$
holds.

(Michael Reitmeir)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.