53^{rd} Austrian Mathematical Olympiad

National Competition—Preliminary Round

30th April 2022

1. Prove that for all real positive numbers x, y and z the double inequality

$$0 < \frac{1}{x+y+z+1} - \frac{1}{(x+1)(y+1)(z+1)} \leq \frac{1}{8}$$

holds.

For which values does equality hold in the right-hand inequality?

(Walther Janous)

2. Points A, B, C and D lie on a circle in this order. Let O be the circle's center. Suppose AC and BD are orthogonal. Let F be the foot of the altitude from O to AB.

Prove that $CD = 2 \cdot OF$.

(Karl Czakler)

3. At each integer on the number line from 0 through 2022, a person is standing at the start of a process.

In each move, two of these people, standing at least two units apart, are chosen. Each of these walks one unit closer to the other.

If no further move is possible, the process ends.

Prove that this process must terminate after a finite number of moves and determine all possible final configurations where the persons can stand. (The configurations only take into account how many persons stand at each number.)

(Birgit Vera Schmidt)

4. Determine all triples (p, q, r) of prime numbers such that $4q - 1$ is a prime number, too, and

$$\frac{p+q}{p+r} = r - p.$$

(Walther Janous)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.