52^{nd} Austrian Mathematical Olympiad

National Competition—Preliminary Round

1st May 2021

1. Let a, b, c be positive real numbers with $a + b + c = 1$.

   Prove that
   $$\frac{a}{2a+1} + \frac{b}{3b+1} + \frac{c}{6c+1} \leq \frac{1}{2}.$$

   When does equality hold?

   (Karl Czakler)

2. Let ABC denote a triangle. The point X lies on the extension of AC beyond A, such that $AX = AB$. Similarly, the point Y lies on the extension of BC beyond B such that $BY = AB$.

   Prove that the circumcircles of ACY and BCX intersect a second time in a point different from C that lies on the bisector of the angle $\angle BCA$.

   (Theresia Eisenkölbl)

3. Let $n \geq 3$ be an integer.

   On a circle, there are n points. Each of them is labelled with a real number at most 1 such that each number is the absolute value of the difference of the two numbers immediately preceding it in clockwise order.

   Determine the maximal possible value of the sum of all numbers as a function of n.

   (Walther Janous)

4. On a blackboard, there are 17 integers not divisible by 17. Alice and Bob play a game. Alice starts and they alternately play the following moves:

   - Alice chooses a number a on the blackboard and replaces it with a^2.
   - Bob chooses a number b on the blackboard and replaces it with b^3.

   Alice wins if the sum of the numbers on the blackboard is a multiple of 17 after a finite number of steps.

   Prove that Alice has a winning strategy.

   (Daniel Holmes)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.