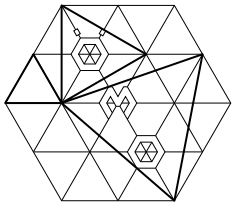

55th Austrian Mathematical Olympiad

Junior Regional Competition

11th June 2024

1. Let x and y be positive real numbers with $x + y = 1$. Prove that

$$\frac{x+1}{y} + \frac{y+1}{x} \geq 6.$$

When does equality hold?

(Karl Czakler)

2. Let $ABCD$ be a trapezoid with parallel sides AB and CD, with $\angle BAD = 90°$ and with $AB + CD = BC$. Furthermore, let M be the mid-point of AD.

Prove that $\angle CMB = 90°$.

(Karl Czakler)

3. Anna, Berta and Clara write the square numbers $1, 4, 9, \ldots, 2025$ on a blackboard, compute their sum and observe that it is divisible by 3. Then, they agree to the following game: In each round, Anna will cross out one number, then Berta will do the same, and then Clara will do the same. This continues until all numbers are crossed out. Clara has the goal that the sum of the remaining numbers after each round is divisible by 3.

   a) Prove that Anna cannot stop Clara from reaching her goal if Clara has Berta's help.

   b) Prove that Berta can stop Clara from reaching her goal even if Clara has Anna's help.

(Richard Henner)

4. Determine the maximal number of consecutive positive integers such that each of these integers has a common divisor with 2024 greater than 1.

(Walther Janous)

Working time: 4 hours.
Each problem is worth 8 points.