54^{th} Austrian Mathematical Olympiad

Junior Regional Competition

13th June 2023

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1. Let x, y, z be nonzero real numbers with

$$\frac{x+y}{z} = \frac{y+z}{x} = \frac{z+x}{y}.$$

Determine all possible values of

$$\frac{(x+y)(y+z)(z+x)}{xyz}.$$

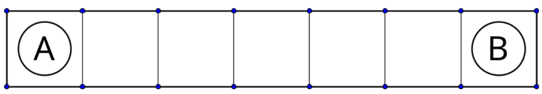
(Walther Janous)

2. Let $ABCDEF$ be a regular hexagon with sidelength s. The points P and Q are on the diagonals BD and DF, respectively, such that $BP = DQ = s$.

Prove that the three points C, P and Q are on a line.

(Walther Janous)

3. Alice and Bob play a game on a strip of $n \geq 3$ squares with two game pieces. At the beginning, Alice's piece is on the first square while Bob's piece is on the last square. The figure shows the starting position for a strip of $n = 7$ squares.



The players alternate. In each move, they advance their own game piece by one or two squares in the direction of the opponent's piece. The piece has to land on an empty square without jumping over the opponent's piece. Alice makes the first move with her own piece. If a player cannot move, they lose.

For which n can Bob ensure a win no matter how Alice plays?

For which n can Alice ensure a win no matter how Bob plays?

(Karl Czakler)

4. Determine all triples (a, b, c) of positive integers such that

$$a! + b! = 2^{c!}.$$

(Walther Janous)

Working time: 4 hours.
Each problem is worth 8 points.