49^{th} Austrian Mathematical Olympiad

Beginners' Competition

12th June 2018

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1. Let a, b and c denote positive real numbers. Prove that

$$\frac{a}{c} + \frac{c}{b} \geq \frac{4a}{a+b}.$$

   When does equality hold?

(Walther Janous)

2. Let ABC be an acute-angled triangle, M the midpoint of the side AC and F the foot on AB of the altitude through the vertex C.

   Prove that $AM = AF$ holds if and only if $\angle BAC = 60°$.

(Karl Czakler)

3. For a given integer $n \geq 4$ we examine whether there exists a table with three rows and n columns which can be filled by the numbers $1, 2, \ldots, 3n$ such that

   - each row totals to the same sum z and
   - each column totals to the same sum s.

   Prove:

   (a) If n is even, such a table does not exist.

   (b) If $n = 5$, such a table does exist.

(Gerhard J. Woeginger)

4. For a positive integer n we denote by $d(n)$ the number of positive divisors of n and by $s(n)$ the sum of these divisors. For example, $d(2018)$ is equal to 4 since 2018 has four divisors (1, 2, 1009, 2018) and $s(2018) = 1 + 2 + 1009 + 2018 = 3030$.

   Determine all positive integers x such that $s(x) \cdot d(x) = 96$.

(Richard Henner)

Working time: 4 hours.
Each problem is worth 8 points.