47^{th} Austrian Mathematical Olympiad

Beginners' Competition

June 16, 2016

1. Determine all nonnegative integers n having two distinct positive divisors with the same distance from $\frac{n}{3}$.

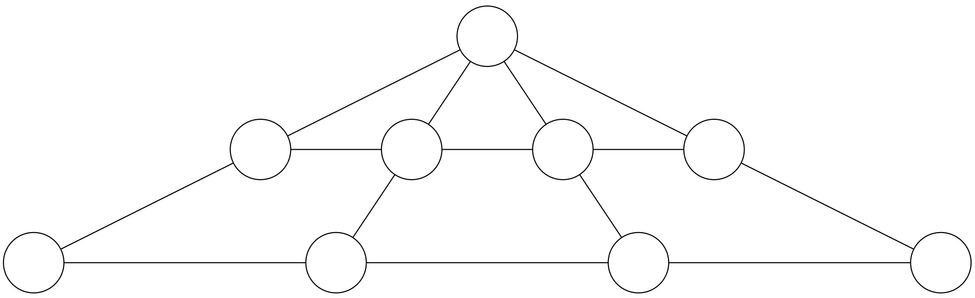
(Richard Henner)

2. Prove that all real numbers $x \neq -1$, $y \neq -1$ with $xy = 1$ satisfy the following inequality:

$$\left(\frac{2+x}{1+x}\right)^2 + \left(\frac{2+y}{1+y}\right)^2 \geq \frac{9}{2}$$

(Karl Czakler)

3. We consider the following figure:



We are looking for labellings of the nine fields with the numbers 1, 2, . . . , 9. Each of these numbers has to be used exactly once. Moreover, the six sums of three resp. four numbers along the drawn lines have to be be equal.
Give one such labelling.
Show that all such labellings have the same number in the top field.
How many such labellings do there exist? (Two labellings are considered different, if they disagree in at least one field.)

(Walther Janous)

4. Let $ABCDE$ be a convex pentagon with five equal sides and right angles at C and D. Let P denote the intersection point of the diagonals AC and BD.
Prove that the segments PA and PD have the same length.

(Gottfried Perz)

Working time: 4 hours.
Each problem is worth 8 points.