53^{rd} Austrian Mathematical Olympiad

Regional Competition

31st March 2022

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1. Let a and b be positive real numbers with $a^2 + b^2 = \frac{1}{2}$. Prove that

$$\frac{1}{1-a} + \frac{1}{1-b} \geq 4.$$

When does equality hold?

(Walther Janous)

2. Determine the number of ten-digit positive integers with the following properties:

   - Each of the digits $0, 1, 2, \ldots, 8$ and 9 is contained exactly once.
   - Each digit, except 9, has a neighbouring digit that is larger than it.

   (*Note.* For example, in the number 1230, the digits 1 and 3 are the neighbouring digits of 2 and 2 and 0 are the neighbouring digits of 3. The digits 1 and 0 have only one neighbouring digit.)

(Karl Czakler)

3. Let ABC denote a triangle with $AC \neq BC$. Let I and U denote the incenter and circumcenter of the triangle ABC, respectively. The incircle touches BC and AC in the points D and E, respectively. The circumcircles of the triangles ABC and CDE intersect in the two points C and P.

   Prove that the common point S of the lines CU and PI lies on the circumcircle of the triangle ABC.

(Karl Czakler)

4. We are given the set

$$M = \{-2^{2022}, -2^{2021}, \ldots, -2^2, -2, -1, 1, 2, 2^2, \ldots, 2^{2021}, 2^{2022}\}.$$

Let T be a subset of M, such that neighbouring numbers have the same difference when the elements are ordered by size.

   (a) Determine the maximum number of elements that such a set T can contain.

   (b) Determine all sets T with the maximum number of elements.

(Walther Janous)

Working time: 4 hours.
Each problem is worth 8 points.