48^{th} Austrian Mathematical Olympiad

Regional Competition (Qualifying Round)

30th March 2017

1. Let x_1, x_2, ..., x_9 be nonnegative real numbers subject to the condition

$$x_1^2 + x_2^2 + \ldots + x_9^2 \geq 25.$$

Prove that there exist three of these numbers with a sum of at least 5.

(Karl Czakler)

2. Let $ABCD$ be a cyclic quadrilateral with perpendicular diagonals and circumcenter O. Let g be the line obtained by reflection of the diagonal AC along the angle bisector of $\angle BAD$.

Prove that the point O lies on the line g.

(Theresia Eisenkölbl)

3. The nonnegative integers 2000, 17 and n are written on the blackboard. Alice and Bob play the following game: Alice begins, then they play in turns. A move consists in replacing one of the three numbers by the absolute difference of the other two. No moves are allowed, where all three numbers remain unchanged. The player who is in turn and cannot make an allowed move loses the game.

- Prove that the game will end for every number n.
- Who wins the game in the case $n = 2017$?

(Richard Henner)

4. Determine all integers $n \geq 2$, satisfying

$$n = a^2 + b^2,$$

where a is the smallest divisor of n different from 1 and b is an arbitrary divisor of n.

(Walther Janous)

Working time: 4 hours.
Each problem is worth 8 points.