# $56^{\text{th}}$ Austrian Mathematical Olympiad

National Competition—Final Round (Day 1)

28th May 2025

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1. Let $k$ and $n$ be positive integers.

   Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

   $$x^k f(y) - y^n f(x) = f\left(\frac{y}{x}\right)$$

   for all real numbers $x$ and $y$ with $x \neq 0$.

2. Let $ABC$ be a triangle with $AC < BC$. Let $L$ be the point of intersection of the angle bisector of $\angle ACB$ with the perpendicular bisector of $AC$. Let $M$ be the midpoint of segment $BC$ and let $N$ be the midpoint of the arc from $A$ to $B$ of the circumcircle which contains $C$.

   Show that $LMN$ is a right triangle.

3. Anna and Bertha play two games on a regular 2025-gon. One of these is the Triangle Game and the other is the Quadrilateral Game. In both games, a move consists of drawing a diagonal of the 2025-gon that has not been drawn previously and does not intersect any previously drawn diagonal in an inner point. The players take alternate moves starting with Anna. The game ends when no additional allowed diagonal can be drawn.

   (a) The Triangle Game: If one or two triangles are created when the diagonal is drawn, the player labels the resulting triangle(s) with her initial. The player with the most labeled triangles at the end of the game wins. If they have the same number of labeled triangles, the game ends in a tie. Does Anna have a winning strategy, does Bertha have a winning strategy, or must the game end in a tie if both players play in an optimal way?

   (b) The Quadrilateral Game: If one or two quadrilaterals with no diagonals inside are created when the diagonal is drawn, the player labels the resulting quadrilateral(s) with her initial. No diagonals may be drawn in labeled quadrilaterals. The player with the most labeled quadrilaterals at the end of the game wins. If they have the same number of labeled quadrilaterals, the game ends in a tie. Does Anna have a winning strategy, does Bertha have a winning strategy, or must the game end in a tie if both players play in an optimal way?

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.

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4. For a positive integer $n$, let $a_1, a_2, \ldots, a_n$ be positive real numbers with $a_1 a_2 \cdots a_n = 2^n$. Prove that
$$a_1^2 + a_1 a_2^2 + a_1 a_2 a_3^2 + \ldots + a_1 a_2 \cdots a_{n-1} a_n^2 \geq 4(2^n - 1),$$
and determine when equality holds.

5. Let $ABC$ be a triangle. For every integer $n \geq 2$, point $D_n$ lies on segment $CB$ with $CD_n = \frac{1}{n} CB$, and point $E_n$ lies on segment $CA$ with $CE_n = \frac{1}{n+1} CA$.

   Prove that all lines $D_n E_n$ pass through a common point.

6. Determine all positive integers $n$, such that there exist positive integers $z_1 < z_2 < \cdots < z_n$ satisfying $\gcd(z_j, z_k) = z_k - z_j$ for all $j$ and $k$ with $1 \leq j < k \leq n$.

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.