# $55^{\text{th}}$ Austrian Mathematical Olympiad

## National Competition—Final Round (Day 1)

29th May 2024

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1. Determine the smallest constant $C$ such that the inequality

$$(X+Y)^2(X^2+Y^2+C)+(1-XY)^2 \geq 0$$

holds for all real numbers $X$ and $Y$.

For which values of $X$ and $Y$ does equality hold for this smallest constant $C$?

*(Walther Janous)*

2. Let $ABC$ be an acute triangle with $AB > AC$. Let $D$, $E$ and $F$ denote the feet of its altitudes on $BC$, $AC$ and $AB$, respectively. Let $S$ denote the intersection of lines $EF$ and $BC$.

Prove that the circumcircles $k_1$ and $k_2$ of the two triangles $AEF$ and $DES$ touch in $E$.

*(Karl Czakler)*

3. Initially, the numbers 1, 2, …, 2024 are written on a blackboard. Trixi and Nana play a game, taking alternate turns. Trixi plays first.

The player whose turn it is chooses two numbers $a$ and $b$, erases both, and writes their (possibly negative) difference $a-b$ on the blackboard. This is repeated until only one number remains on the blackboard after 2023 moves. Trixi wins if this number is divisible by 3, otherwise Nana wins.

Which of the two has a winning strategy?

*(Birgit Vera Schmidt)*

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.

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4. Let $ABC$ be an obtuse triangle with orthocenter $H$ and centroid $S$. Let $D$, $E$ and $F$ be the midpoints of segments $BC$, $AC$, $AB$, respectively.

   Show that the circumcircle of triangle $ABC$, the circumcircle of triangle $DEF$ and the circle with diameter $HS$ have two distinct points in common.

   *(Josef Greilhuber)*

5. Let $n$ be a positive integer and let $z_1$, $z_2$, ..., $z_n$ be positive integers such that for $j = 1, 2, \ldots, n$ the inequalities
$$z_j \leq j$$
   hold and $z_1 + \ldots + z_n$ is even.

   Prove that the number 0 occurs among the values of
$$z_1 \pm z_2 \pm \ldots \pm z_n,$$
   where $+$ or $-$ can be chosen independently for each operation.

   *(Walther Janous)*

6. For each prime number $p$, determine the number of residue classes modulo $p$ which can be represented as $a^2 + b^2$ modulo $p$, where $a$ and $b$ are arbitrary integers.

   *(Daniel Holmes)*

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.