# $54^{\text{th}}$ Austrian Mathematical Olympiad

National Competition—Final Round (Day 1)

24th May 2023

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1. Let $\alpha$ be a nonzero real number.

   Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ with

   $$f(f(x+y)) = f(x+y) + f(x)f(y) + \alpha xy$$

   for all $x$, $y \in \mathbb{R}$.

   *(Walther Janous)*

2. Let $ABC$ be a triangle, and $O$ its circumcenter. The circumcircle of triangle $AOC$ shall intersect the segment $BC$ in points $C$ and $D$ and the segment $AB$ in points $A$ and $E$.

   Prove that triangles $BDE$ and $AOC$ have equal circumradii.

   *(Karl Czakler)*

3. Alice and Bob play a game, in which they take turns drawing segments of length 1 in the Euclidean plane. Alice begins, drawing the first segment, and from then on, each segment must start at the endpoint of the previous segment. It is not permitted to draw the segment lying over the preceding one. If the new segment shares at least one point - except for its starting point - with one of the previously drawn segments, one has lost.

   a) Show that both Alice and Bob could force the game to end, if they don't care who wins.

   b) Is there a winning strategy for one of them?

   *(Michael Reitmeir)*

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.

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25th May 2023

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4. Written on a blackboard are the 2023 numbers

$$2023, 2023, \dots, 2023.$$

The numbers on the blackboard are now modified, in a sequence of moves. In each move, two numbers on the blackboard—call them $x$ and $y$—are chosen, deleted, and replaced by the single number $\frac{x+y}{4}$. Such moves are carried out until there is only one number left on the blackboard.

Prove that this number is always greater than 1.

*(Walther Janous)*

5. Let $ABC$ be an acute triangle, with $AC \neq BC$. Let $M$ be the midpoint of segment $AB$. Let $H$ be the orthocenter of triangle $ABC$, $D$ the footpoint of the altitude through $A$ on $BC$ and $E$ the footpoint of the altitude through $B$ on $AC$.

Prove that lines $AB$, $DE$ and the orthogonal to $MH$ through $C$ intersect in a point $S$.

*(Karl Czakler)*

6. Determine whether there exists a real number $r$ such that the equation

$$x^3 - 2023x^2 - 2023x + r = 0$$

has three different rational solutions.

*(Walther Janous)*

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.