49^{th} Austrian Mathematical Olympiad

National Competition (Final Round, part 2, first day)

31st May 2018

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1. Let $\alpha \neq 0$ be a real number.

   Find all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with

   $$f(f(x)+y) = \alpha x + \frac{1}{f\left(\frac{1}{y}\right)}$$

   for all x, $y \in \mathbb{R}_{>0}$.

   (Walther Janous)

2. Let A, B, C and D be four different points lying on a common circle in this order. Assume that the line segment AB is the (only) longest side of the inscribed quadrilateral $ABCD$.

   Prove that the inequality

   $$AB + BD > AC + CD$$

   holds.

   (Karl Czakler)

3. There are n children in a room. Each child has at least one piece of candy. In Round 1, Round 2, etc., additional pieces of candy are distributed among the children according to the following rule:

   In Round k, each child whose number of pieces of candy is relatively prime to k receives an additional piece.

   Show that after a sufficient number of rounds the children in the room have at most two different numbers of pieces of candy.

   (Theresia Eisenkölbl)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.

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1st June 2018

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4. Let ABC be a triangle and P a point inside the triangle such that the centers M_B and M_A of the circumcircles k_B and k_A of triangles ACP and BCP, respectively, lie outside the triangle ABC. In addition, we assume that the three points A, P and M_A are collinear as well as the three points B, P and M_B. The line through P parallel to side AB intersects circles k_A and k_B in points D and E, respectively, where $D, E \neq P$.

   Show that $DE = AC + BC$.

   (Walther Janous)

5. On a circle 2018 points are marked.

   Each of these points is labeled with an integer. Let each number be larger than the sum of the preceding two numbers in clockwise order.

   Determine the maximal number of positive integers that can occur in such a configuration of 2018 integers.

   (Walther Janous)

6. Determine all digits z such that for each integer $k \geq 1$ there exists an integer $n \geq 1$ with the property that the decimal representation of n^9 ends with at least k digits z.

   (Walther Janous)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.