47^{th} Austrian Mathematical Olympiad

National Competition (Final Round, part 2, first day)

May 25, 2016

1. Let $\alpha \in \mathbb{Q}^+$. Determine all functions $f\colon \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f\left(\frac{x}{y} + y\right) = \frac{f(x)}{f(y)} + f(y) + \alpha x$$

holds for all x, $y \in \mathbb{Q}^+$.

Here, $\mathbb{Q}^+$ denotes the set of positive rational numbers.

(Walther Janous)

2. Let ABC be a triangle. Its incircle meets the sides BC, CA and AB in the points D, E and F, respectively. Let P denote the intersection point of ED and the line perpendicular to EF and passing through F, and similarly let Q denote the intersection point of EF and the line perpendicular to ED and passing through D.

Prove that B is the mid-point of the segment PQ.

(Karl Czakler)

3. Consider arrangements of the numbers 1 through 64 on the squares of an 8×8 chess board, where each square contains exactly one number and each number appears exactly once.

A number in such an arrangement is called *super-plus-good*, if it is the largest number in its row and at the same time the smallest number in its column.

Prove or disprove each of the following statements:

(a) Each such arrangement contains at least one super-plus-good number.

(b) Each such arrangement contains at most one super-plus-good number.

(Gerhard J. Woeginger)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.

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May 26, 2016

4. Let a, b, $c \geq -1$ be real numbers with $a^3 + b^3 + c^3 = 1$. Prove that
$$a + b + c + a^2 + b^2 + c^2 \leq 4.$$
When does equality hold?

(Karl Czakler)

5. Consider a board consisting of $n \times n$ unit squares where $n \geq 2$. Two cells are called neighbors if they share a horizontal or vertical border. In the beginning, all cells together contain k tokens. Each cell may contain one or several tokens or none.

   In each turn, choose one of the cells that contains at least one token for each of its neighbors and move one of those to each of its neighbors. The game ends if no such cell exists.

   (a) Find the minimal k such that the game does not end for any starting configuration and choice of cells during the game.

   (b) Find the maximal k such that the game ends for any starting configuration and choice of cells during the game.

(Theresia Eisenkölbl)

6. Let a, b, c be integers such that
$$\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a}$$
is an integer.

   Prove that each of the numbers
$$\frac{ab}{c}, \frac{ac}{b} \quad \text{and} \quad \frac{bc}{a}$$
is an integer.

(Gerhard J. Woeginger)

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