54th Austrian Mathematical Olympiad

Regional Competition—Solutions

30th March 2023

Problem 1. *Let a, b and c be real numbers with $0 \leq a, b, c \leq 2$. Prove that*

$$(a-b)(b-c)(a-c) \leq 2.$$

When does equality hold?

(Karl Czakler)

Solution. We order the variables by size:

For $a \geq b \geq c$, all three factors are positive and we have $(a-b)(b-c)(a-c) \geq 0$.

For $b \geq c \geq a$ and $c \geq a \geq b$, two of the factors are negative and one factor is positive, so we have again $(a-b)(b-c)(a-c) \geq 0$.

For all the other orderings of variables, we have either three negative factors or one negative and two positive factors. This implies $(a-b)(b-c)(a-c) \leq 0$, so the inequality holds for these cases and there is no case of equality.

Let us now consider $a \geq b \geq c$.

With the AM-GM-inequality, we get

$$(a-b)(b-c) \leq \frac{(a-b+b-c)^2}{4} = \frac{(a-c)^2}{4}.$$

So we obtain

$$(a-b)(b-c)(a-c) \leq \frac{(a-c)^2}{4}(a-c) = \frac{(a-c)^3}{4} \leq \frac{2^3}{4} = 2.$$

The two remaining cases of orderings can be treated analogously.

We see that equality holds for $a-c=2$ and $a-b=b-c$, which implies $a=2$, $b=1$ and $c=0$. Taking into account the analogous cases, we see that equality holds exactly for the triples $(2,1,0), (1,0,2)$ and $(0,2,1)$.

(Karl Czakler) □

Problem 2. *Let $ABCD$ be a rhombus with $\angle BAD < 90°$. The circle passing through D with center A intersects the line CD a second time in point E. Let S be the intersection of the lines BE and AC.*

Prove that the points A, S, D and E lie on a circle.

(Karl Czakler)

Solution. By the inscribed angle theorem, it is enough to show that $\angle SED = \angle SAD$.

Since $ABCD$ is a rhombus, we have

$$\angle SAD = \frac{1}{2}\angle BAD.$$

Since $ABCE$ is an isosceles trapezoid, we have by symmetry that

$$\angle SED = \angle ECS = \frac{1}{2}\angle DCB = \frac{1}{2}\angle BAD,$$

which finishes the proof.

(Theresia Eisenkölbl) □

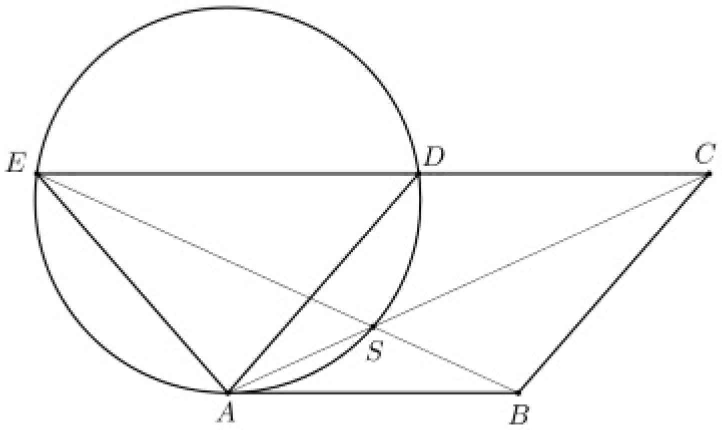

Figure 1: Problem 2

Problem 3. *Determine all natural numbers $n \geq 2$ with the property that there are two permutations $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ of the numbers $1, 2, \ldots, n$ such that $(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$ are consecutive natural numbers.*

(Walther Janous)

Answer. The permutations exist if and only if n is odd.

Solution. We have

$$(a_1 + b_1) + (a_2 + b_2) + \ldots + (a_n + b_n) = 2(1 + 2 + \ldots + n) = n(n + 1).$$

On the other hand, there is a natural number N such that

$$a_1 + b_1 = N, a_2 + b_2 = N + 1, \ldots, a_n + b_n = N + n - 1$$

and therefore

$$(a_1 + b_1) + (a_2 + b_2) + \ldots + (a_n + b_n) = nN + (1 + \ldots + (n - 1)) = nN + n(n - 1)/2.$$

We obtain the equation $n(n + 1) = nN + n(n - 1)/2$ which becomes $N = n + 1 - \frac{n-1}{2} = \frac{n+3}{2}$. Therefore, the number N is an integer if and only if n is odd.

It remains to investigate if two permutations with the desired property exist for every odd number n with $n \geq 3$. Let $n = 2k + 1$ with $k \geq 1$.

Experimenting with $k = 1$ and $k = 2$ can lead to the following pattern:

$$\begin{pmatrix} 1 & k+2 & 2 & k+3 & 3 & \ldots & 2k+1 & k+1 \\ k+1 & 1 & k+2 & 2 & k+3 & \ldots & k & 2k+1 \end{pmatrix}$$

Summing the two rows gives the $2k+1$ consecutive numbers $k+2, k+3, \ldots, 3k+1, 3k+2$ as desired.

(Walther Janous) □

Problem 4. *Determine all pairs (x, y) of positive integers such that for $d = \gcd(x, y)$ the equation*

$$xyd = x + y + d^2$$

holds.

(Walther Janous)

Answer. There are three such pairs, $(x, y) = (2, 2)$, $(x, y) = (2, 3)$ and $(x, y) = (3, 2)$.

Solution. For $x = 1$, we get $d = 1$ and the given equation becomes the contradiction $y = y + 2$. This works analogously for $y = 1$.

Therefore, we can assume $x \geq 2$ and $y \geq 2$.

We start with the case $d = 1$ which gives the equation

$$xy = x + y + 1 \iff (x-1)(y-1) = 2.$$

The possible factorizations $2 = 1 \cdot 2$ and $2 = 2 \cdot 1$ give the pairs $(x, y) = (2, 3)$ and $(x, y) = (3, 2)$, respectively, because $\gcd(x, y) = 1$ is satisfied.

Now, we treat the case $d \geq 2$. The given equation is equivalent to

$$\frac{1}{xd} + \frac{1}{yd} + \frac{d}{xy} = 1.$$

Because of $xd \geq 4$ and $yd \geq 4$, we get

$$1 \leq \frac{1}{4} + \frac{1}{4} + \frac{d}{xy} \iff xy \leq 2d.$$

Together with $xy \geq d^2$, we obtain $d = 2$, $x = y = 2$ which gives indeed the third pair $(x, y) = (2, 2)$ with $\gcd(2, 2) = 2$.

(Walther Janous) □