53^{rd} Austrian Mathematical Olympiad

National Competition—Final Round—Solutions

25th/26th May 2022

Problem 1. *Find all functions* $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ *with*

$$a - f(b) \mid af(a) - bf(b) \text{ for all } a, b \in \mathbb{Z}_{>0}.$$

(Theresia Eisenkölbl)

Answer. The only solution is the identity $f(x) = x$ for all $x \in \mathbb{Z}_{>0}$.

Solution. For $a = f(b)$, we immediately get that $af(a) - bf(b) = 0$.

Therefore,

$$f(b)(f(f(b)) - b) = 0,$$

and we have $f(f(b)) = b$ for all $b \in \mathbb{Z}_{>0}$.

Now, we replace b with $f(b)$ in the given relation and obtain

$$a - b \mid af(a) - bf(b) = (a - b)f(a) + b(f(a) - f(b)), .$$

from which we obtain

$$a - b \mid b(f(a) - f(b)).$$

For $b = 1$, we get

$$a - 1 \mid f(a) - f(1).$$

If we replace a with $f(a)$, we get

$$f(a) - 1 \mid a - f(1).$$

This implies that for all $a > f(1)$, we have $f(a) - 1 \leq a - f(1)$. If we had $f(1) > 1$, this would imply $f(a) < a$ for $a > f(1)$ and therefore either $a = f(f(a)) < f(a) < a$, which is impossible, or $f(a) \leq f(1)$, which cannot hold for infinitely many a because of $f(f(a)) = a$. Therefore, we have $f(1) = 1$ and $a - 1 = f(a) - 1$, so that f has to be the identity. The identity clearly satisfies the given relation, so it is the only solution.

(Theresia Eisenkölbl) □

Problem 2. *Let ABC be an acute, scalene triangle with orthocenter H, and let M be the midpoint of segment AB, and w the angular bisector of angle $\angle ACB$. Let S be the intersection of w and the perpendicular bisector of AB, and F the foot of the altitude from H onto w.*

Prove that segments MS and MF are of equal length.

(Karl Czakler)

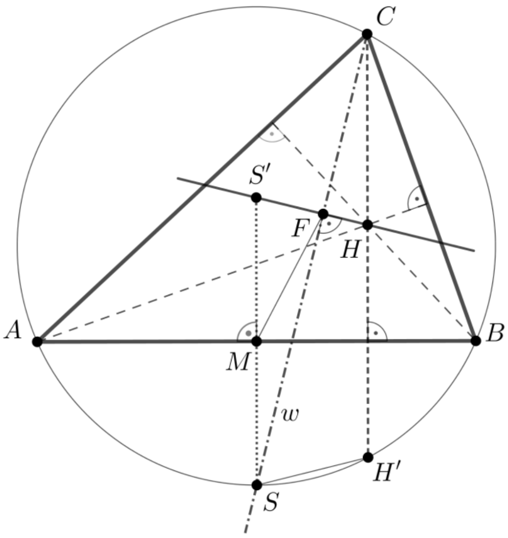
Solution. As usual, we will label the angles in the triangle at A, B and C with α, β and γ, respectively.

It is well-known that S as well as the reflection of H in the line AB are on the circumcircle of the triangle ABC. We label the reflection of H with H'.

In the following, we will work with oriented angles modulo 180°.

We have $\angle SCH' = \frac{\gamma}{2} - (90° - \beta)$. Using the angle sum in the right triangle FHC, we obtain $\angle FHH' = \beta + \frac{\gamma}{2}$.

On the other hand, the inscribed angle theorem for the circumcircle of ABC gives $\angle HH'S = \angle CH'S = \angle CBS = \beta + \angle ABS = \beta + \angle ACS = \beta + \frac{\gamma}{2}$.



By definition of H', the reflection in AB maps H to H'. Since the angles $\angle HH'S$ and $\angle H'HF$ coincide up to orientation, the reflection also maps the line $H'S$ to the line HF.

Let S' be the image of S with respect to this reflection. Because of the previous observation, S' must be on the line HF. Therefore, the triangle $FS'S$ is a right triangle. This clearly implies that M is the mid-point of the segment SS' and is therefore the circumcenter of triangle $FS'S$. Thales' theorem implies $MS = MF$, as desired.

(Theresia Eisenkölbl, Josef Greilhuber) □

Problem 3. *Lisa writes a positive integer in the decimal system on a board and repeats the following steps:*

The last digit is deleted from the number on the board and then four times the deleted digit is added to the remaining shorter number (or to 0 *if the original number was a single digit). The result of this calculation is now the new number on the board.*

This is repeated until the first time she gets a number that has already been on the board.

(a) *Show that the sequence of steps always terminates.*

(b) *What is the last number on the board if Lisa starts with the number* $53^{2022} - 1$*?*

Example: If Lisa starts with the number 2022*, she gets* $202 + 4 \times 2 = 210$ *in the first step and then subsequently*

$$2022 \mapsto 210 \mapsto 21 \mapsto 6 \mapsto 24 \mapsto 18 \mapsto 33 \mapsto 15 \mapsto 21.$$

Since Lisa gets 21 *a second time, she stops.*

(Stephan Pfannerer)

Solution. Let f be the map defined by the given operation on the number. We can write a positive integer x uniquely as $10a + b$ with $a, b \in \mathbb{Z}_{\geq 0}$ and $0 \leq b \leq 9$, and get $f(x) = f(10a + b) = a + 4b$. We denote with $(x_i)_{i \geq 0}$ the sequence of numbers obtained by Lisa if she starts with the positive integer x_0. The sequence is defined by the recursion $x_{i+1} = f(x_i)$.

(a) We first note that it is immediate by definition that $x_i \in \mathbb{Z}_{>0}$ for all $i \geq 0$ and the process is therefore well-defined.

Now, we show that for $x_i \geq 40$, the next element in the sequence is smaller, i.e. $x_i > x_{i+1}$. We write again $x_i = 10a + b$ as above. From $10a + b = x_i > x_{i+1} = a + 4b$, we obtain the equivalent inequality $9a > 3b$ which is certainly true because $a > 3$ and $b \leq 9$.

Next, we prove that for $x_i \leq 39$, we also have $x_{i+1} \leq 39$. This immediately follows from $x_{i+1} = f(x_i) = f(10a+b) = a + 4b \leq 3 + 4 \cdot 9 = 39$.

Therefore, there is a number $N \geq 0$, such that $x_i \in \{1, 2, \ldots, 39\}$ for all $i \geq N$. These are just finitely many possible values, so there are $i \neq j$ with $x_i = x_j$.

(b) The answer is 39. First, we observe that $f(x) \equiv 4 \cdot x \pmod{39}$: Let $x = 10a + b$, as before. Then we get:

$$f(x) = a + 4b \equiv 40a + 4b = 4 \cdot (10a + b) = 4x \pmod{39}.$$

We calculate the residue of the starting number modulo 39. We have

$$53^{2022} - 1 \equiv 1^{2022} - 1 \equiv 0 \pmod{13}$$

and similarly

$$53^{2022} - 1 \equiv (-1)^{2022} - 1 \equiv 0 \pmod{3},$$

so we get $39 | 53^{2022} - 1 = x_0$. Using the above observation, we conclude that $39 | x_i$ for all $i \geq 0$.

Let N be the smallest index with $0 < x_N < 40$. Since $39 | x_N$, we must have $x_N = 39$. Since x_i is strictly decreasing for $i < N$ and $f(39) = 39$, the number 39 is the first one written twice on the blackboard.

(Michael Drmota) □

Problem 4. *Decide if for every polynomial P of degree ≥ 1 with integer coefficients, there are infinitely many primes that each divide a $P(n)$ for a positive integer n.*

(Walther Janous)

Answer. There are infinitely many such primes for every polynomial satisfying the conditions.

Solution. We write $P(x) = a_m x^m + \ldots + a_1 x + a_0$ with $m \geq 1$, $a_m \neq 0$ and integers a_j, $0 \leq j \leq m$.

- If $a_0 = 0$, we have $p \mid P(p)$ for every prime p.
- If $a_0 \neq 0$, we assume that there are only finitely many primes with the desired property. We label them $p_1, \ldots, p_N$ (we have $N \geq 1$, because the non-constant polynomial cannot take the values ± 1 for all positive integers).

  Let q be the product of these N primes. Then, we have for all positive integers k that

  $$P(a_0 q^k) = a_m (a_0 q^k)^m + \ldots + a_1 a_0 q^k + a_0$$
  $$\iff P(a_0 q^k) = a_0 (a_m a_0^{m-1} q^{km} + \ldots + a_1 q^k + 1).$$

  The expression in parentheses is clearly not divisible by any of the N primes. Therefore, it has to take the values ± 1, and we get

  $$P(a_0 q^k) = \pm a_0.$$

  As before, we can argue that the non-constant polynomial P cannot take just two values for infinitely many arguments. This contradiction implies the existence of infinitely many primes with the desired property.

(Walther Janous) □

Problem 5. *Let ABC be an isosceles triangle with base AB.*

We choose an interior point P of the altitude in C. The circle with diameter CP intersects the line connecting B and P a second time in D_P and the line connecting points A and C a second time in E_P.

Prove that there exists a point F, such that for every choice of P the points D_P, E_P and F are collinear.

(Walther Janous)

Answer. The point F with this property is the mid-point of AB.

Solution. Let M be the mid-point of AB. We want to prove that M is on all lines $g_P = D_PE_P$ and therefore, the desired point F.

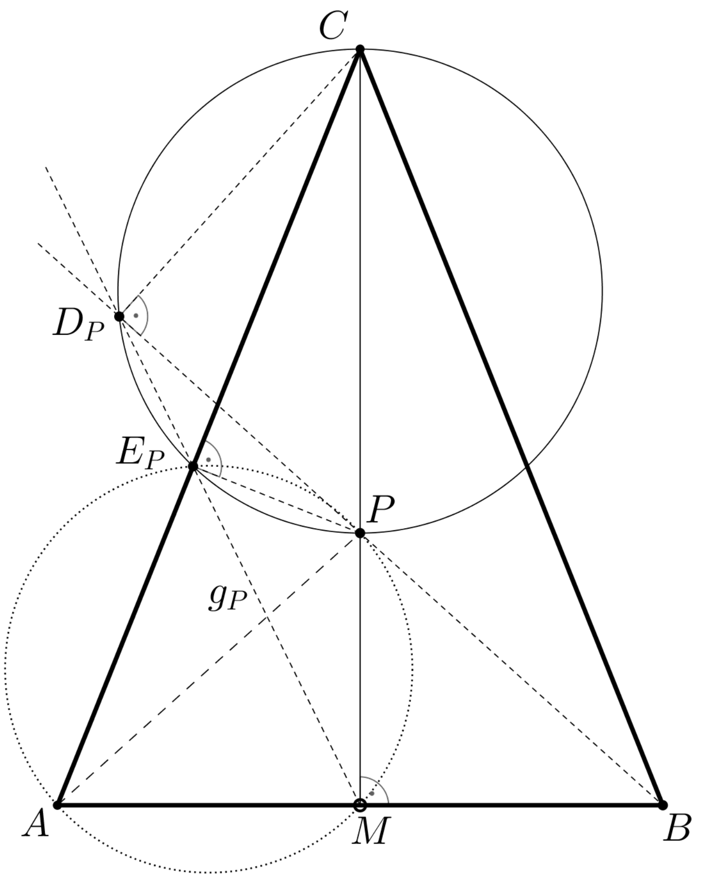


Figure 1: Problem 5

The points C, D_P, E_P and P lie on a circle by definition of D_P and E_P. By Thales' theorem, we get $PE_P \perp AE_P$ and by definition of M, we get $PM \perp AM$. We obtain that $AMPE_P$ is also a cyclic quadrilateral.

In the circumcircle of CD_PE_PP, we compute

$$\angle(D_PE_P, AC) = \angle(D_PE_P, E_PC) = \angle(D_PP, PC) = \angle(BP, MC)$$

and in the circumcircle of $AMPE_P$, we compute

$$\angle(ME_P, AC) = \angle(ME_P, AE_P) = \angle(MP, AP) = \angle(MC, AP).$$

Since MC is the altitude of the isosceles triangle, we have $\angle(BP, MC) = \angle(MC, AP)$, and we obtain $\angle(D_PE_P, AC) = \angle(ME_P, AC)$.

Therefore, the points D_P, E_P and M are collinear independent of P, and M is the desired point F.

(Theresia Eisenkölbl) □

Problem 6. *(a) Prove that a square with sidelength* 1000 *can be tiled with* 31 *squares such that at least one of them has sidelength smaller than* 1.

(b) Prove that there is also a tiling with 30 *squares with the same properties.*

(Walther Janous)

Solution. (a) We first divide the square into four squares with half the side-length. Then, we choose one of them and divide it again into four smaller squares which gives a tiling with 7 squares. We apply this method 10 times in total, each of which adds 3 squares to the number of squares in the tiling, so we get a tiling with $1 + 10 \cdot 3 = 31$ squares. The four smallest have a side-length of $1000/2^{10} < 1$.

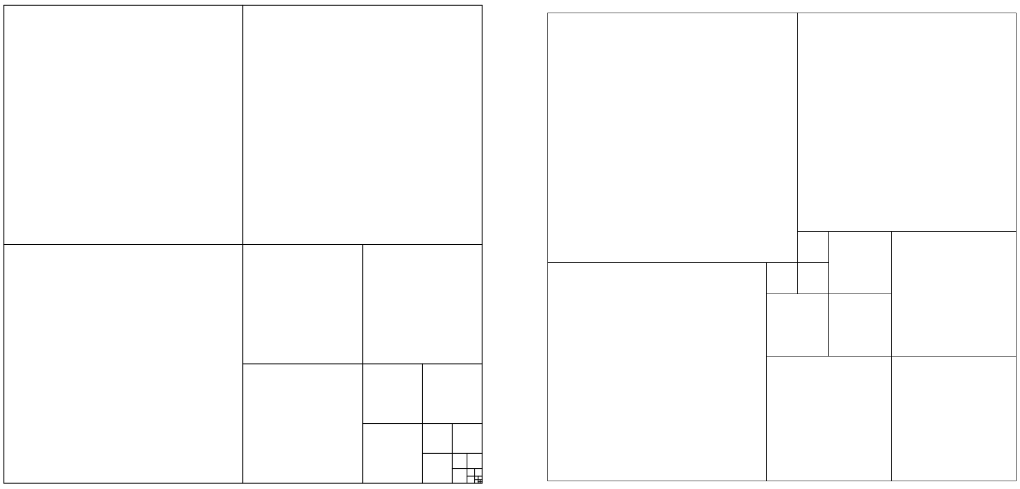


Figure 2: On the left, we see the tiling for question (a), on the right, we see the construction for question (b) for a square with side-length 15.

(b) To be able to work with integer coordinates, we scale the 1000×1000-square to a 1023×1023-square and we will scale the whole tiling back at the end. We divide the square into a 512×512-square, two 511×511-squares and a region R_1 which is a 512×512-square Q_1, where a 1×1-square has been removed. Therefore, we have used three squares and still have to tile the region R_1.

Now, we divide Q_1 into four smaller squares which divides the region R_1 into three squares of sidelength 252 plus a region R_2 which is a 256×256-square Q_2 where a 1×1-square has been removed.

The eighth iteration of this argument will add three 2×2-squares plus a region R_8, which gives a total of $3 \cdot 9 = 27$ squares. Since R_8 can be divided in three 1×1-squares, we have a tiling of the 1023×1023-square into 30 squares where the three smallest have side-length 1. Scaling the whole figure back to side-length 1000 gives a tiling of the desired type with the smallest side-length $1000/1023$.

(Walther Janous) □