Czech-Polish-Slovak-Austrian Match, Day 1

1. Find all quadruples (a, b, c, d) of positive integers satisfying $\gcd(a, b, c, d) = 1$ and

$$a \mid b + c, \quad b \mid c + d, \quad c \mid d + a, \quad d \mid a + b.$$

2. In an acute triangle ABC, the incircle ω touches BC at D. Let I_a be the excenter of ABC opposite to A, and let M be the midpoint of DI_a. Prove that the circumcircle of triangle BMC is tangent to ω.

3. For any two convex polygons P_1 and P_2 with mutually distinct vertices, denote by $f(P_1, P_2)$ the total number of their vertices that lie on a side of the other polygon. For each positive integer $n \geq 4$, determine

$$\max\{f(P_1, P_2) \mid P_1 \text{ and } P_2 \text{ are convex } n\text{-gons}\}.$$

(We say that a polygon is *convex* if all its internal angles are strictly less than $180°$.)

Czech-Polish-Slovak-Austrian Match, Day 2

4. Determine the number of 2021-tuples of positive integers such that the number 3 is an element of the tuple and consecutive elements of the tuple differ by at most 1.

5. The sequence $a_1, a_2, a_3, \ldots$ satisfies $a_1 = 1$, and for all $n \geq 2$, it holds that

$$a_n = \begin{cases} a_{n-1} + 3 & \text{if } n-1 \in \{a_1, a_2, \ldots, a_{n-1}\}; \\ a_{n-1} + 2 & \text{otherwise.} \end{cases}$$

Prove that for all positive integers n, we have

$$a_n < n \cdot (1 + \sqrt{2}).$$

6. Let ABC be an acute triangle and suppose points A, A_b, B_a, B, B_c, C_b, C, C_a, and A_c lie on its perimeter in this order. Let $A_1 \neq A$ be the second intersection point of the circumcircles of triangles AA_bC_a and AA_cB_a. Analogously, $B_1 \neq B$ is the second intersection point of the circumcircles of triangles BB_cA_b and BB_aC_b, and $C_1 \neq C$ is the second intersection point of the circumcircles of triangles CC_aB_c and CC_bA_c. Suppose that the points A_1, B_1, and C_1 are all distinct, lie inside the triangle ABC, and do not lie on a single line. Prove that lines AA_1, BB_1, CC_1, and the circumcircle of triangle $A_1B_1C_1$ all pass through a common point.

CPSA 2021 – solutions

1. Find all quadruples (a, b, c, d) of positive integers satisfying $\gcd(a, b, c, d) = 1$ and

$$a \mid b + c, \quad b \mid c + d, \quad c \mid d + a, \quad d \mid a + b.$$

Vítězslav Kala (Czech Republic)

Solution. Without loss of generality, assume that $a = \max\{a, b, c, d\}$. Then

$$a \leq b + c \leq 2a,$$

and so we have 2 possible cases:

CASE 1. $b + c = 2a$.

In this case, $a = b = c$, and so

$$a = c \mid (d + a) - a = d.$$

But $a \geq d$, and so we must have $a = b = c = d$, and by the coprimality assumption, we get the solution $(1, 1, 1, 1)$.

CASE 2. $b + c = a$.

Let $c + d = kb$ for some positive integer k. The relation $c \mid d + a$ then implies

$$c \mid c + d + (b + c) = (k + 1)b + c,$$

and so $(k + 1)b = lc$ for some positive integer l.

Moreover, we have

$$d \mid a + b = 2b + c,$$

and so $md = m(kb - c) = 2b + c$ for some positive integer m.

We now have the system

$$(k + 1)b = lc$$
$$(km - 2)b = (m + 1)c.$$

From the second equation, we see that $km - 2 > 0$, and so

$$l = (k + 1)\frac{b}{c} = \frac{(k + 1)(m + 1)}{km - 2} = 1 + \frac{k + m + 3}{km - 2}.$$

Since l is an integer, we have

$$km - 2 \mid k + m + 3.$$

However, if $(k - 1)(m - 1) > 6$, then $km - 2 > k + m + 3$, which would contradict $km - 2 \mid k + m + 3$.

Note that $m = 1$ is not possible, for then we would have $d = 2b+c > b+c = a$. Therefore, there are 5 remaining cases.

CASE 2a. $k = 1$.

Then $m - 2 \mid m + 4$, and so $m - 2 \mid 6$, i.e. m is one of the numbers $3, 4, 5, 8$ (recall that we know that $m - 2 = km - 2 > 0$). The corresponding values of l are then $8, 5, 4, 3$. By the coprimality assumption, this uniquely determines the solutions as

$$(a, b, c, d) \in \{(5, 4, 1, 3), (7, 5, 2, 3), (3, 2, 1, 1), (5, 3, 2, 1)\}.$$

CASE 2b. $k = 2$ for $2 \le m \le 7$.

We then have $2m - 2 \mid m + 5$, and so the only possibilities are $(m, l) \in \{(3, 3), (7, 2)\}$, to which the corresponding solutions are

$$(a, b, c, d) \in \{(2, 1, 1, 1), (5, 2, 3, 1)\}.$$

CASE 2c. $m = 2$ for $3 \le k \le 7$.

Then $2k - 2 \mid k + 5$, and so the only possibilities are $(k, l) \in \{(3, 3), (7, 2)\}$ and so

$$(a, b, c, d) \in \{(7, 3, 4, 5), (5, 1, 4, 3)\}.$$

CASE 2d. $k = 3$ for $3 \le m \le 4$.

Then $3m - 2 \mid m + 6$. The only possibility is $m = 4$, and so $l = 2$ and

$$(a, b, c, d) = (3, 1, 2, 1).$$

CASE 2e. $k = 4$ and $m = 3$.

Here, we get $l = 2$ and

$$(a, b, c, d) = (7, 2, 5, 3).$$

Altogether, we have found the following solutions:

$$(a, b, c, d) \in \{(7, 5, 2, 3), (7, 3, 4, 5), (7, 2, 5, 3), (5, 4, 1, 3), (5, 3, 2, 1),$$
$$(5, 2, 3, 1), (5, 1, 4, 3), (3, 2, 1, 1), (3, 1, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1)\}$$

(and all their rotations when we remove the assumption that a is largest).

Alternative solution. We will show a different finish of case 2 from the previous solution. we need to fulfill

$$b \mid c + d, \qquad c \mid d + b, \qquad d \mid 2b + c.$$

We will distinguish cases based on which of the numbers b, c, d is largest.

CASE 2a. $b = \max(b, c, d)$.

Then $b = (c+d)/2$ or $b = c+d$ by the same reasoning as in in case 1. The first statement yields $b = c = d$, which gives $(a, b, c, d) = (2, 1, 1, 1)$ when combined with the coprimality condition.

If, on the other hand, $b = c + d$ holds, then the conditions above ensure $c \mid 2d$ and $d \mid 3c$, thus $c \mid 2d \mid 6c$, so that $2d \in \{c, 2c, 3c, 6c\}$. Each of these possibilities determines a and b uniquely by the coprimality condition. We arrive at the solutions

$$(a, b, c, d) \in \{(3, 2, 1, 1), (5, 3, 2, 1), (5, 4, 1, 3), (7, 5, 2, 3)\}.$$

CASE 2b. $c = \max(b, c, d)$.

By the same reasoning as in case 2a, we get

$$(a, b, c, d) \in \{(2, 1, 1, 1), (5, 2, 3, 1), (5, 1, 4, 3), (7, 2, 5, 3), (3, 1, 2, 1)\}.$$

CASE 2c. $d = \max(b, c, d)$.

Because of $b + c = a \geq d \geq (2b + c)/3$, we can only have $d \mid 2b + c$ for $d = (2b + c)/2$ or $d = (2b + c)/3$. Again, the latter case yields $b = c = d$ and the solution $(2, 1, 1, 1)$.

For $d = (2b + c)/2 = b + c/2$, we find that c has to be even, and so $c = 2C$ for a positive integer C. Now, we obtain $b \mid c + d = 2C + (b + C)$, which means $b \mid 3C$, as well as $C \mid c \mid d + b = (b + C) + b$, and therefore $C \mid 2b$. We infer $b \mid 3C \mid 6b$ and from that $3C \in \{b, 2b, 3b, 6b\}$. Only $3C = 2b$ yields a new solution, specifically $(7, 3, 4, 5)$.

2. In an acute triangle ABC, the incircle ω touches BC at D. Let I_a be the excenter of ABC opposite to A, and let M be the midpoint of DI_a. Prove that the circumcircle of triangle BMC is tangent to ω. *Patrik Bak (Slovakia)*

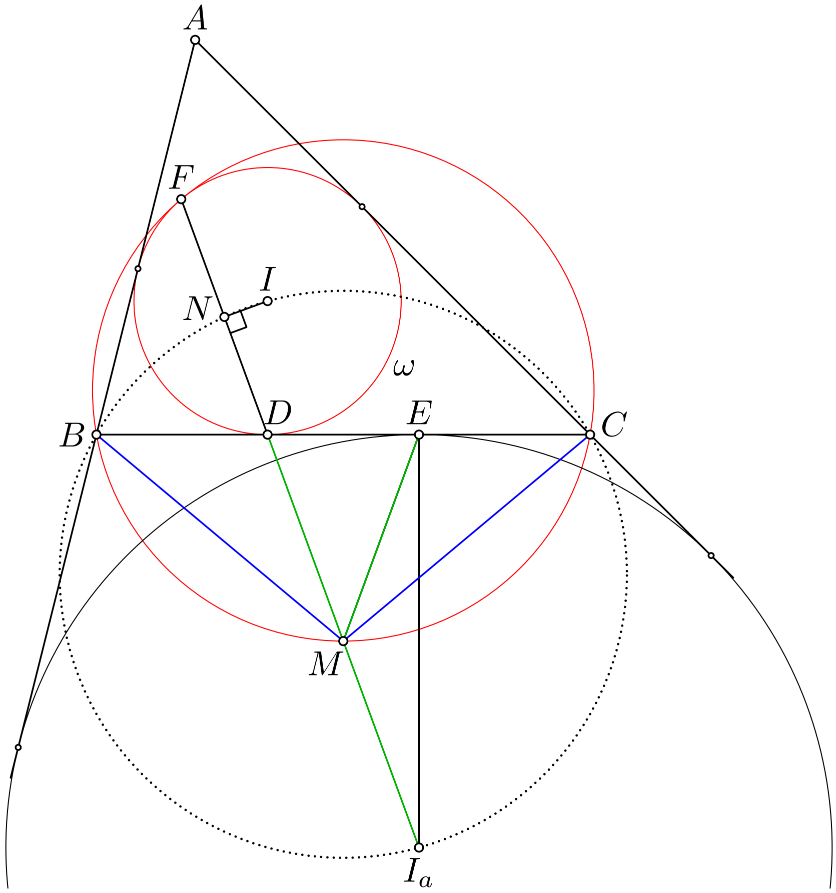
Solution. Let I be the incenter of ABC and let F be the second intersection point of MD and ω and let N be the midpoint of FD. Points B, N, I, C, I_a are concyclic, as they lie on the circle with a diameter II_a. The power of a point gives

$$DF \cdot DM = 2DN \cdot \frac{1}{2}DI_a = DB \cdot DC,$$

which means that F, B, M, C are concyclic.

Let E the projection of I_a on BC. It is well-known that D and E are symmetric with respect to the midpoint of BC. Since $MD = MI_a = ME$, the congruence of triangles MDB and MEC gives $MB = MC$.

It remains to realize that the circle through F, B, M, and C is tangent to ω at F. This can be seen from homothety: Without loss of generality let BC be horizontal. Then D is a lowest point of ω, Since $MB = MC$, also M is the lowest point of the circumricle of MBC. If there is a circle through M, B, C tangent to ω, then the tangency point



must be the second intersection point of the circumcircle of MBC and line MD, which is indeed F.

3. For any two convex polygons P_1 and P_2 with mutually distinct vertices, denote by $f(P_1, P_2)$ the total number of their vertices that lie on a side of the other polygon. For each positive integer $n \geq 4$, determine

$$\max\{f(P_1, P_2) \mid P_1 \text{ and } P_2 \text{ are convex } n\text{-gons}\}.$$

(We say that a polygon is *convex* if all its internal angles are strictly less than 180°.)

Josef Tkadlec (Czech Republic)

Solution. We will show that $F(n) = \lfloor 4n/3 \rfloor$ for any $n \geq 4$.

For the construction, see Figure 1.

For the bound, fix $n \geq 4$ and two convex n-gons P_1, P_2. Call any of the $2n$ vertices *good* if it lies on a side of the other polygon.

If the interiors of P_1 and P_2 do not intersect, then at most 2 points are good. Indeed, in this case, there is a line ℓ such that each corresponding (closed) half-plane contains one

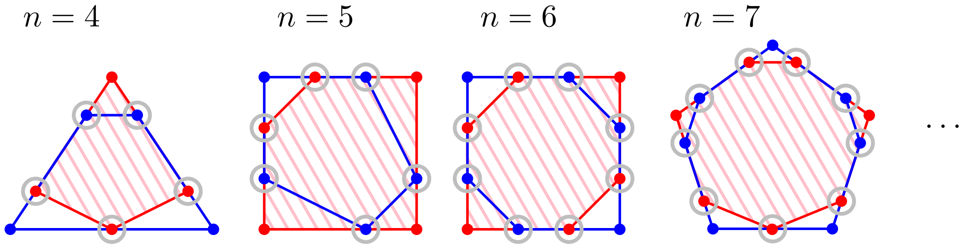


Figure 1

of the polygons. Since both the polygons are convex, at most $2+2$ of their vertices lie on ℓ, hence $f(P_1, P_2) \le 2 < \lfloor 4n/3 \rfloor$ for any $n \ge 4$.

Suppose that the interiors do intersect and take any point X inside both P_1 and P_2, not lying on any line through two vertices. A rotating ray emanating from X defines a cyclical order of the $2n$ vertices. It suffices to show that among any three consecutive vertices in this order, at most 2 are good – the bound then follows by summing over all consecutive triples.

Color vertices of P_1 black (B) and vertices of P_2 white (W). By symmetry, it suffices to distinguish three cases of the colors of the three consecutive vertices: WWW, WWB, and WBW.

Split the plane into n "wedges" with a shared apex X and rays passing through all (black) vertices of P_1. Note that since X is inside P_1, these wedges are convex and each wedge contains precisely one side of P_1 (and each side of P_1 is contained in precisely one wedge).

In each of the three cases, we suppose that all three vertices are good, argue that the three vertices in fact lie on the same line and then reach a contradiction with the convexity of P_2 by finding three collinear white vertices on that line (see Figure 2).

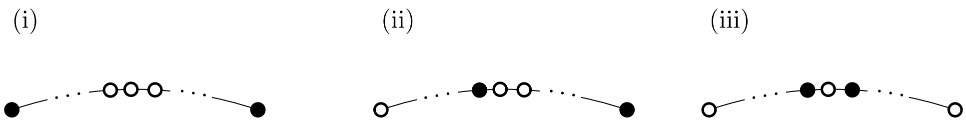


Figure 2

(i) WWW: All three white vertices lie in the same wedge, hence on the same side of P_1, a contradiction.

(ii) BWW: Both white vertices lie in the same wedge, hence on the same side of P_1. This side has the black vertex as one endpoint, hence the three vertices are collinear. Since the black vertex is also good, there is a third white vertex on that line on the other side of the black vertex, a contradiction.

(iii) BWB: The white vertex lies on the segment connecting the two black vertices, hence the three vertices are collinear. Since both the black vertices are also good, there is one more white vertex on each side, a contradiction.

This completes the proof.

Remark. One can show that $F(3) = 3 \neq \lfloor 4 \cdot 3/3 \rfloor$.

4. Determine the number of 2021-tuples of positive integers such that the number 3 is an element of the tuple and consecutive elements of the tuple differ by at most 1.

Walther Janous (Austria)

Solution. First, we count the number of such tuples ignoring the first property.

Any tuple $(a_1, \dots, a_{2021})$ having the second property is uniquely determined by $\min_{i=1}^{2021} a_i$ and the tuple $(a_2 - a_1, \dots, a_{2021} - a_{2020}) \in \{-1, 0, 1\}^{2020}$.

Hence, if the minimum is given, there are 3^{2020} tuples satisfying only the second condition.

To account for the first condition, that is, to have 3 as an entry of the sequence, we need the minimum from above to belong to $\{1, 2, 3\}$ on one hand, and the maximum of all a_i to be greater than or equal to 3 on the other hand. This is equivalent to $\min_{i=1}^{2021} a_i \in \{1, 2, 3\}$ and for the sequence $(a_1, \dots, a_{2021})$ to have entries from $\{1, 2\}$ (these sequences all have the second property).

Therefore, the desired number of sequences fulfilling both conditions is $3 \cdot 3^{2020} - 2^{2021} = 3^{2021} - 2^{2021}$.

5. The sequence $a_1, a_2, a_3, \dots$ satisfies $a_1 = 1$, and for all $n \geq 2$, it holds that

$$a_n = \begin{cases} a_{n-1} + 3 & \text{if } n - 1 \in \{a_1, a_2, \dots, a_{n-1}\}; \\ a_{n-1} + 2 & \text{otherwise.} \end{cases}$$

Prove that for all positive integers n, we have

$$a_n < n \cdot (1 + \sqrt{2}).$$

Dominik Burek (Poland)

Solution. First, it is easy to see that for any $n \geq 2$, we have $a_n = 2n + k - 1$ where $k = \max\{i \colon a_i < n\}$. Indeed, a_n is obtained by adding to a_1 twos and threes in $n - 1$ steps, where 3 is added in steps $a_1, a_2, \dots, a_k$, and two is added in the remaining $n - 1 - k$ steps. Hence $a_n = a_1 + 3k + 2(n - k - 1) = 2n + k - 1$. Also, note that such a k satisfies $k + 1 < n$ provided $n \geq 3$.

Now, we shall prove the following stronger statement: For any $n \geq 1$, we have

$$(1 + \sqrt{2})n - 2 < a_n < (1 + \sqrt{2})n.$$

This is clearly true for $n = 1$ and $n = 2$. For the inductive step, let $n \geq 3$ and suppose that this holds for all indices smaller than n. Write $a_n = 2n + k - 1$ where $k = \max\{i \colon a_i < n\}$, so that we have $a_k < n \leq a_{k+1}$. We have to prove that

$$(1 + \sqrt{2})n - 2 < 2n + k - 1 < (1 + \sqrt{2})n.$$

This is equivalent to

$$(\sqrt{2} - 1)n - 1 < k < (\sqrt{2} - 1)n + 1.$$

We have

$$(\sqrt{2} - 1)n - 1 \leq (\sqrt{2} - 1)a_{k+1} - 1 < (\sqrt{2} - 1)(\sqrt{2} + 1)(k + 1) - 1 = k,$$

where the first inequality holds since $n \leq a_{k+1}$, and the second one holds by the induction hypothesis applied to $k + 1$.

Similarly,

$$\begin{aligned}(\sqrt{2} - 1)n + 1 > (\sqrt{2} - 1)a_k + 1 &> (\sqrt{2} - 1)((\sqrt{2} + 1)k - 2) + 1 \\ &= k - 2(\sqrt{2} - 1) + 1 = k + 3 - 2\sqrt{2} > k\end{aligned}$$

because $n > a_k$ and, by the inductive hypothesis, $a_k > (1 + \sqrt{2})k - 2$.

This finishes the proof of the double inequality

$$(1 + \sqrt{2})n - 2 < a_n < (1 + \sqrt{2})n,$$

and we are done.

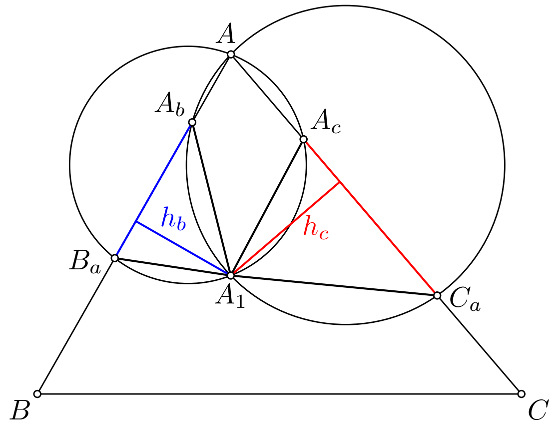
6. Let ABC be an acute triangle and suppose points A, A_b, B_a, B, B_c, C_b, C, C_a, and A_c lie on its perimeter in this order. Let $A_1 \neq A$ be the second intersection point of the circumcircles of triangles AA_bC_a and AA_cB_a. Analogously, $B_1 \neq B$ is the second intersection point of the circumcircles of triangles BB_cA_b and BB_aC_b, and $C_1 \neq C$ is the second intersection point of the circumcircles of triangles CC_aB_c and CC_bA_c. Suppose that the points A_1, B_1, and C_1 are all distinct, lie inside the triangle ABC, and do not lie on a single line. Prove that lines AA_1, BB_1, CC_1, and the circumcircle of triangle $A_1B_1C_1$ all pass through a common point.

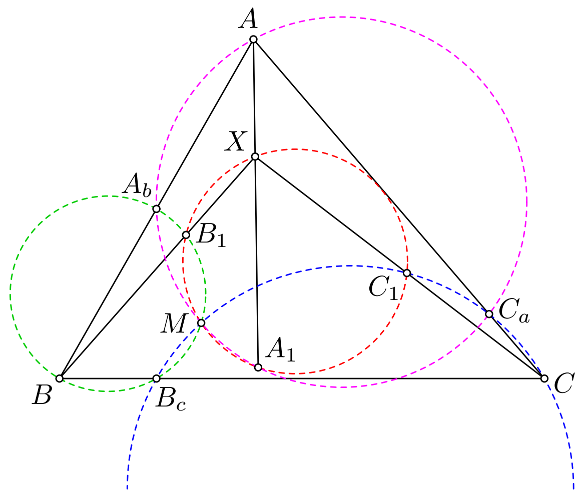
Josef Tkadlec (Czech Republic), Patrik Bak (Slovakia)

Solution. First, we prove will that the three lines are concurrent.

Point A_1 is the center of the spiral similarity that maps B_aA_b to A_cC_a, and so the triangles $A_1B_aA_b$ and $A_1A_cC_a$ are similar (this is also easy to verify by direct angle-chasing). We aim to use the trigonometric form Ceva's Theorem. Let h_c and h_b be the distances of A_1 to the sides AB and AC, respectively. Using the similar triangles, we get

$$\frac{\sin \sphericalangle BAA_1}{\sin \sphericalangle A_1AC} = \frac{h_c/AA_1}{h_b/AA_1} = \frac{h_c}{h_b} = \frac{B_aA_b}{A_cC_a},$$





which is cyclic in terms of A, B, C, hence the lines are concurrent by the trigonometric form Ceva's Theorem

Next, we will prove that the intersection point of lines AA_1, BB_1 and CC_1 lines on the circumcircle of $A_1B_1C_1$.

Focus on points B_1, B_c and C_1 lying on the lines determined by the vertices of XBC. Due to Miquel's theorem, the circumcircles of XB_1C_1, BB_1B_c and CC_1B_c are concurrent, denote their common point by M.

Applying Miquel's theorem on points A_b, B_c, and C_a lying on the sides of ABC gives that M lies also on the circumcircle of AA_bC_a. Due to this, we can repeat the logic used to define M with regards to triangle XAB to prove that X, A_1, B_1 and M are concyclic. Altogether, points X, A_1, B_1, C_1 and M are concyclic, so we are done.