52^{nd} Austrian Mathematical Olympiad

Regional Competition—Solutions

25th March 2021

Problem 1. *Let a and b be positive integers and c be a positive real number satisfying*

$$\frac{a+1}{b+c} = \frac{b}{a}.$$

Prove that $c \geq 1$ holds.

(Karl Czakler)

Solution. The following equations are equivalent to the given equation:

$$\begin{aligned} a^2 + a &= b^2 + bc \\ 4a^2 + 4a + 1 &= 4b^2 + 4bc + 1 \\ (2a+1)^2 &= 4b^2 + 4bc + 1. \end{aligned}$$

Assume to the contrary that $c < 1$ holds. This yields

$$(2b)^2 = 4b^2 < (2a+1)^2 = 4b^2 + 4bc + 1 < 4b^2 + 4b + 1 = (2b+1)^2.$$

This is a contradiction as the square of an integer cannot lie strictly between two consecutive square numbers. Therefore, $c \geq 1$ holds and, for instance, $a = b$ yields $c = 1$ and therefore there is a solution of the equation with $c \geq 1$.

(Karl Czakler) □

Problem 2. *Let ABC be an isosceles triangle with $AC = BC$ and circumcircle k. The point D lies on the shorter arc of k over the chord BC and is different from B and C. Let E denote the intersection of CD and AB.*

Prove that the line through B and C is a tangent of the circumcircle of the triangle BDE.

(Karl Czakler)

Solution. We denote the center of the circumcircle of the triangle BDE by M and $\angle BAC = \angle CBA$ by α. Since the quadrilateral $ABDC$ is cyclic, we obain $\angle BDE = \alpha$. By the inscribed angle theorem, $\angle BME = 2\alpha$ and thus $\angle EBM = \angle MEB = 90^\circ - \alpha$. Therefore,

$$180^\circ = \angle CBA + \angle MBC + \angle EBM = \alpha + \angle MBC + 90^\circ - \alpha = \angle MBC + 90^\circ$$

holds and we get

$$\angle MBC = 90^\circ,$$

completing the proof.

(Karl Czakler) □

Problem 3. *The numbers $1, 2, \ldots, 2020$ und 2021 are written on a blackboard. The following operation is executed:*
Two numbers are chosen, both are erased and replaced by the absolute value of their difference.
This operation is repeated until there is only one number left on the blackboard.

(a) *Show that 2021 can be the final number on the blackboard.*

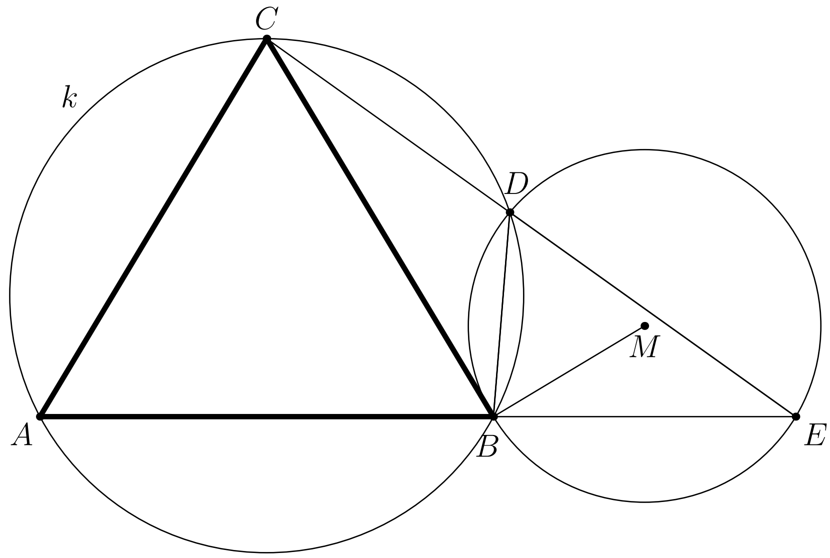

Figure 1: Problem 2

(b) *Show that* 2020 *cannot be the final number on the blackboard.*

(Karl Czakler)

Solution. (a) Let us first choose the following 1010 pairs of numbers:

$$(1, 2);\ (3, 4);\ \ldots;\ (2019, 2020).$$

The absolute value of the difference within each of these pairs is 1. After applying the operation for each of these pairs, the number 2021 and 1010 times the number 1 remain on the blackboard. Now we execute the given operation 505 times with pairs of the form $(1, 1)$. Then the number 2021 and 505 times the number 0 remain on the blackboard. As $2021 - 0 = 2021$ and $0 - 0 = 0$, we end up with 2021 as the final number on the board after additional 505 operations, regardless of the pairs we pick at each step. *Remark.* Here we have just given one example among several possible choices of building pairs and ending up with 2021.

(b) We prove a more general statement: The final remaining number on the blackboard cannot be even.

As

$$a - b \equiv a + b \pmod 2,$$

we obtain that the parity of the sum of all numbers on the board is an invariant throughout the game. At the beginning, the sum of the numbers on the blackboard is

$$\frac{2021 \cdot 2022}{2} = 2021 \cdot 1011,$$

an odd number. Therefore, the final number on the board must be odd as well. In particular, 2020 cannot be the final number on the blackboard.

(Karl Czakler) □

Problem 4. *Determine all triples* (x, y, z) *of positive integers satisfying*

$$x \mid (y + 1), \quad y \mid (z + 1) \ \text{ and } \ z \mid (x + 1).$$

(Walther Janous)

Answer. There are ten triples satisfying the three conditions. They are given by $(1,1,1)$, $(1,1,2)$, $(1,3,2)$, $(3,5,4)$ and their cyclic permutations.

Solution. For the sake of readability, we use the notation $a \mid b+c$ instead of $a \mid (b+c)$ throughout the proof.

Without loss of generality, let x be the smallest of the three numbers (or one of the smallest), i.e. $x \le y$ and $x \le z$. From $z \mid x+1$ we obtain $x \le z \le x+1$. Thus we have to consider two cases.

- Case 1. Let $z = x$. Then $z = x \mid x+1$ leads to $x = z = 1$ and $y \mid z+1 = 2$. Therefore $y = 1$ or $y = 2$, and we get the two solutions $(1,1,1)$ and $(1,2,1)$.
- Case 2. Let $z = x+1$. Then the two conditions $x \mid y+1$ and $y \mid x+2$ must be fulfilled. In particular, we obtain $x \le y+1$ and $y \le x+2$. This yields $x-1 \le y \le x+2$ and we have to examine the following cases for y.
  - Case 2a. Let $0 < y = x$. The conditions $x \mid x+1$ and $x \mid x+2$ can only hold simultaneously for $x = 1$, giving the solution $(1,1,2)$.
  - Case 2b. Let $y = x+1$. Then the two conditions are $x \mid x+2$ and $x+1 \mid x+2$. They cannot hold simultaneously.
  - Case 2c. Let $y = x+2$. The condition $y = x+2 \mid x+2 = z+1$ is trivially fulfilled. The requirement $x \mid y+1 = x+3$ can only hold for $x \mid 3$. And, indeed, for either $x = 1$ or $x = 3$ the condition is fulfilled and we obtain the solutions $(1,3,2)$ and $(3,5,4)$.

Summing up, the triples $(1,1,1)$, $(1,2,1)$, $(1,1,2)$, $(1,3,2)$ and $(3,5,4)$ fulfill all three conditions.

As each of the three numbers can be the minimum, every cyclic permutation of these triples is a solution as well.

(Lukas Donner) □