52^{nd} Austrian Mathematical Olympiad

Regional Competition

25th March 2021

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1. Let a and b be positive integers and c be a positive real number satisfying

$$\frac{a+1}{b+c} = \frac{b}{a}.$$

Prove that $c \geq 1$ holds.

(Karl Czakler)

2. Let ABC be an isosceles triangle with $AC = BC$ and circumcircle k. The point D lies on the shorter arc of k over the chord BC and is different from B and C. Let E denote the intersection of CD and AB.

Prove that the line through B and C is a tangent of the circumcircle of the triangle BDE.

(Karl Czakler)

3. The numbers $1, 2, \ldots, 2020$ und 2021 are written on a blackboard. The following operation is executed:
Two numbers are chosen, both are erased and replaced by the absolute value of their difference.
This operation is repeated until there is only one number left on the blackboard.

   (a) Show that 2021 can be the final number on the blackboard.

   (b) Show that 2020 cannot be the final number on the blackboard.

(Karl Czakler)

4. Determine all triples (x, y, z) of positive integers satisfying

$$x \mid (y+1), \quad y \mid (z+1) \text{ and } z \mid (x+1).$$

(Walther Janous)

Working time: 4 hours.
Each problem is worth 8 points.