Czech-Polish-Slovak Match

IST Austria, 23 – 26 June 2019

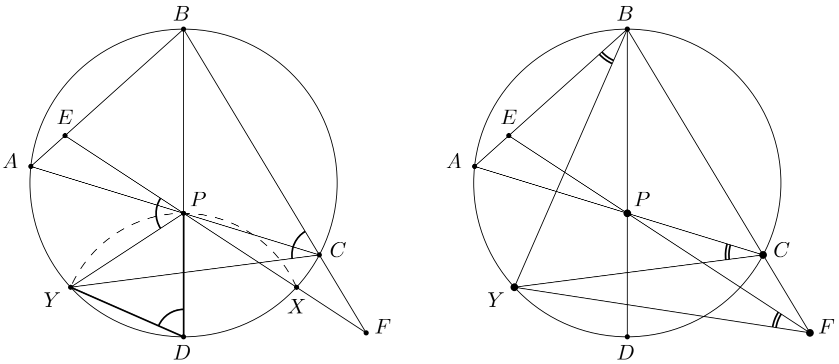
(First day – 24 June 2019)

1. Let ω be a circle. Points A, B, C, X, D, Y lie on ω in this order such that BD is its diameter and $DX = DY = DP$, where P is the intersection of AC and BD. Denote by E, F the intersections of line XP with lines AB, BC, respectively. Prove that points B, E, F, Y lie on a single circle. (*Patrik Bak, Slovakia*)

Solution. First, we show that the quadrilateral $YPCF$ is cyclic. Indeed, by simple angle-chasing we have

$$\angle YPF = 2 \cdot \angle YPD = 180^\circ - \angle BDY = 180^\circ - \angle BCY = \angle YCF.$$

The rest is angle-chasing again. We have $\angle EFY = \angle PFY = \angle PCY = \angle ACY = \angle ABY = \angle EBY$ as desired.



2. We consider positive integers n having at least six positive divisors. Let the positive divisors of n be arranged in a sequence $(d_i)_{1 \le i \le k}$ with

$$1 = d_1 < d_2 < \cdots < d_k = n \quad (k \geq 6).$$

Find all positive integers n such that

$$n = d_5^2 + d_6^2.$$

(*Walther Janous, Austria*)

Solution. In what follows we shall show that this question has the unique answer $n = 500$. Indeed, from $n = d_5^2 + d_6^2$ we readily infer that n has to be even. (For, otherwise d_5 and d_6 had to be odd. This in turn would yield n even.) Therefore $d_2 = 2$ is fixed. Furthermore from $d_5 \mid n$ we get $d_5 \mid d_6^2$ and similarly $d_6 \mid d_5^2$. This means:

Every prime dividing d_5 also divides d_6 and vice versa.

If d_5 has only one prime factor, i.e. it is a power of a prime, then $d_5 = p^k$ and $d_6 = p^{k+1}$. But since $p^k < 2p^k \leq p^{k+1}$, it follows that $p = 2$ and $n = d_5^2 + d_6^2 = 2^{2k} + 2^{2k+2} = 5 \cdot 2^{2k}$. Therefore either $n = 20$, which is not a solution, or

$$d_2 = 2, \qquad d_3 = 4, \qquad d_4 = 5, \qquad d_5 = 8, \qquad d_6 = 10,$$

a contradiction.

Now d_5 and d_6 have at least two prime factors p and q with $p < q$ and $p^2q^2 \mid d_5^2 + d_6^2 = n$. Then $d_5 \geq pq$ and since $1 < p < q, p^2 < pq$ we also have $d_5 \leq pq$. Now

$$d_2 = p = 2, \qquad \{d_3, d_4\} = \{q, p^2\} = \{q, 4\}, \qquad d_5 = pq = 2q, \qquad d_6 = p^2q = 4q.$$

We get $n = d_5^2 + d_6^2 = 20q^2$, hence $q \leq 5$. Checking the cases $q = 3$ and $q = 5$ gives the unique solution $n = 500$.

3. A dissection of a convex polygon into finitely many triangles by segments is called a *trilateration* if no three vertices of the created triangles lie on a single line (vertices of some triangles might lie inside the polygon). We say that a trilateration is *good* if its segments can be replaced with one-way arrows in such a way that the arrows along every triangle of the trilateration form a cycle and the arrows along the whole convex polygon also form a cycle. Find all $n \geq 3$ such that the regular n-gon has a good trilateration. (*Josef Greilhuber, Austria*)

Solution. We show that the regular n-gon has a good trilateration if and only if $3 \mid n$.

Given a regular n-gon and its good trilateration, color the triangles whose arrows go clockwise in black and the other ones in white. In this way, any two triangles sharing an edge have received different colors and all the triangles sharing an edge with the perimeter of the whole n-gon have received the same color (wlog black). We say that a segment in the trilateration is *interior* if it is not one of the sides of the n-gon. Let x be the number of interior segments. Since each interior segment is a side of precisely one white triangle and the sides of white triangle are all different interior segments, we have $3 \mid x$. Arguing likewise for the black triangles, we obtain $3 \mid x + n$. Hence $3 \mid n$.

It remains to show that when $3 \mid n$ then the regular n-gon has a good trilateration. This is straightforward by mathematical induction.