49th Austrian Mathematical Olympiad

Regional Competition (Qualifying Round)—Solutions

5th April 2018

Problem 1. *Let a and b be nonnegative real numbers satisfying $a+b<2$.*

Prove the inequality

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}\le\frac{2}{1+ab}$$

and determine all a and b yielding equality.

(Gottfried Perz)

Solution. We have

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}=\frac{2+a^2+b^2}{(1+a^2)(1+b^2)}=1+\frac{1-a^2b^2}{(1+a^2)(1+b^2)}.$$

Via the Cauchy-Schwarz inequality we get $(1+a^2)(1+b^2)\ge(1+ab)^2$. Furthermore, the arithmetic-geometric means inequality yields $ab\le\left(\frac{a+b}{2}\right)^2<1$. Therefore, the inequality

$$1+\frac{1-a^2b^2}{(1+a^2)(1+b^2)}\le 1+\frac{1-a^2b^2}{(1+ab)^2}=\frac{2}{1+ab}$$

follows and we are done. Finally, equality occurs if and only if $0\le a=b<1$.

(Karl Czakler) □

Problem 2. *Let k be a circle with radius r and AB a chord of k such that $AB>r$. Furthermore, let S be the point on the chord AB satisfying $AS=r$. The perpendicular bisector of BS intersects k in the points C and D. The line through D and S intersects k for a second time in point E.*

Show that the triangle CSE is equilateral.

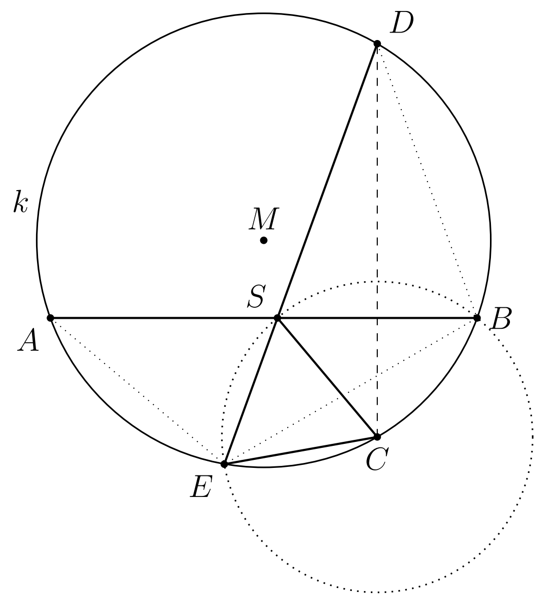
(Stefan Leopoldseder)

Solution. In the first part we prove $CS=CE$, afterwards we will prove $\angle SCE=60°$.

Since C and D lie on the symmedian of BS, $CBDS$ is a kite with

$$CS=CB \qquad \text{and} \qquad \angle CDB=\angle SDC=\angle EDC.$$

The line segments CB and CE therefore have equal length (equal inscribed angles at D). Hence, we have shown that $CS=CB=CE$. Furthermore there exists a circle k_1 with C containing E, S and B.



The triangle ESA is isosceles with base ES, because it is similar to the isosceles triangle BSD with base BS ($\angle SEA = \angle DEA = \angle DBA = \angle DBS$ and $\angle EAS = \angle EAB = \angle EDB = \angle SDB$ follow from the inscribed angle theorem in k). We therefore have $AE = AS = r$, that is, the length of the chord AE is equal to the radius r of the circle k. The central angle for the chord AE is therefore 60°, the inscribed angle $\angle ABE$ is $60/2 = 30°$. But now $\angle SBE = \angle ABE = 30°$ is an inscribed angle in k_1 for the chord SE, hence the central angle is $\angle SCE = 2 \cdot 30 = 60°$.

(Stefan Leopoldseder) □

Problem 3. *Let $n \geq 3$ be a natural number.*

Determine the number a_n of all subsets of $\{1, 2, \ldots, n\}$ consisting of three elements such that one of them is the arithmetic mean of the other two.

(Walther Janous)

Solution. Let $\{a, b, c\}$ be a subset of $M = \{1, 2, \ldots, n\}$ such that $a < b < c$. Then $b = (a + c)/2$ is an element of M if and only if a and c have the same parity.

- Let $n = 2k$ be even. Then the sets $\{a, c\}$ are the $k(k-1)/2$ subsets of M consisting of the k even as well of the k odd numbers in M. Therefore, the desired number of subsets $\{a, b, c\}$ of M equals $a_{2k} = k(k-1)$.
- If $n = 2k + 1$ is odd, we similarly get $k(k-1)/2$ subsets $\{a, c\}$ consisting of two even numbers a and c as well as $(k+1)k/2$ such subsets with odd numbers. Thus, $a_{2k+1} = k^2$.

The two results can be summarized as $a_n = \lfloor \frac{n}{2} \rfloor \cdot \lfloor \frac{n-1}{2} \rfloor$.

(Gerd Baron) □

Problem 4. *Let $d(n)$ be the number of all positive divisors of a natural number $n \geq 2$.*

Determine all natural numbers $n \geq 3$ such that

$$d(n-1) + d(n) + d(n+1) \leq 8.$$

(Richard Henner)

Solution. For even numbers $k \geq 6$ we have $d(k) \geq 4$, since 1, 2, $\frac{k}{2}$, k are four different divisors. It is clear that $n = 3$ is a solution, whereas $n = 5$ is not. For odd numbers $n \geq 7$ we have

$$d(n-1) + d(n) + d(n+1) \geq 4 + d(n) + 4 > 8.$$

From now on, let n be even. If the number $k \geq 6$ is divisible by 3, we have $d(k) \geq 3$, since 1, 3, k are three different divisors. We check that $n = 4$ and $n = 6$ satisfy the condition. If $n \geq 8$ and $n - 1$ is divisible by 3, then

$$d(n-1) + d(n) + d(n+1) \geq 3 + 4 + d(n+1) > 8.$$

If, on the other hand, $n \geq 8$ and $n + 1$ is divisible by 3, then we have

$$d(n-1) + d(n) + d(n+1) \geq d(n-1) + 4 + 3 > 8.$$

Since among the three successive integers $n-1$, n, $n+1$ one has to be divisible by 3, the only remaining case is that n is divisible by 6. In this case n has the six different divisors 1, 2, 3, $\frac{n}{3}$, $\frac{n}{2}$, n, i. e. $d(n) \geq 6$ for $n \geq 12$. Thus, we get

$$d(n-1) + d(n) + d(n+1) \geq d(n-1) + 6 + d(n+1) > 8.$$

Hence, there are no further solutions.

(Gerhard Kirchner) □