49^{th} Austrian Mathematical Olympiad

Beginners' Competition—Solutions

12th June 2018

Problem 1. *Let a, b and c denote positive real numbers. Prove that*

$$\frac{a}{c} + \frac{c}{b} \geq \frac{4a}{a+b}.$$

When does equality hold?

(Walther Janous)

Solution. (Gerhard Kirchner) Multiplication by bc and completing squares yields

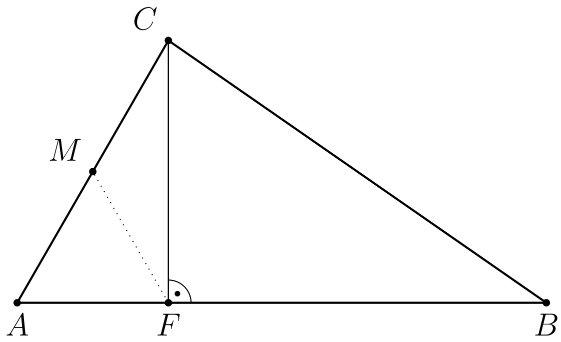
$$\left(c - \frac{2ab}{a+b}\right)^2 + ab\left(\frac{a-b}{a+b}\right)^2 \geq 0.$$

Equality holds if $a = b$ and $c = \frac{2ab}{a+b}$, i. e. for $a = b = c$. □

Problem 2. *Let ABC be an acute-angled triangle, M the midpoint of the side AC and F the foot on AB of the altitude through the vertex C.*
Prove that $AM = AF$ holds if and only if $\angle BAC = 60°$.

(Karl Czakler)

Solution. (Karl Czakler) We will prove the two directions of the equivalence separately.



- Let $AM = AF$.

  Since $\triangle ACF$ is rectangular, the Thales theorem gives $AM = MF$. Hence the triangle AMF is equilateral and therefore $\angle BAC = 60°$.

- Now let $\angle BAC = 60°$.

  Since $\triangle ACF$ is rectangular, we again have $AM = MF$. Now $\triangle AMF$ is isosceles and the base angle is $\angle BAC = \angle FAM = 60°$. Thus, the other base angle $\angle AFM$ also equals $60°$. Hence all angles are $60°$. Therefore, $\triangle AMF$ is equilateral and we have $AM = AF$. □

Problem 3. *For a given integer $n \geq 4$ we examine whether there exists a table with three rows and n columns which can be filled by the numbers 1, 2, ..., $3n$ such that*

- *each row totals to the same sum z and*
- *each column totals to the same sum s.*

Prove:

(a) *If n is even, such a table does not exist.*

(b) *If $n = 5$, such a table does exist.*

(Gerhard J. Woeginger)

Solution. (Gerhard Kirchner)

1. Summing up all entries we get

$$1 + 2 + \ldots + 3n = \frac{3n(3n+1)}{2} = 3z = ns.$$

   Hence $s = \frac{3(3n+1)}{2}$. If n is even, then $3n + 1$ and hence also $3(3n + 1)$ are odd. Therefore s is not an integer, a contradiction.

2. For $n = 5$ we get $s = 24$ and $z = 40$. For example the following table fulfills the conditions:

15	6	2	7	10
8	4	13	12	3
1	14	9	5	11

□

Problem 4. *For a positive integer n we denote by $d(n)$ the number of positive divisors of n and by $s(n)$ the sum of these divisors. For example, $d(2018)$ is equal to 4 since 2018 has four divisors (1, 2, 1009, 2018) and $s(2018) = 1 + 2 + 1009 + 2018 = 3030$.*
Determine all positive integers x such that $s(x) \cdot d(x) = 96$.

(Richard Henner)

Solution. (Clemens Heuberger) We note that $d(1)s(1) = 1$. For all $x \geq 2$, we have $d(x) \geq 2$ and $s(x) \geq x$. Thus $d(x)s(x) = 96$ implies that $2x \leq 96$ and thus $x \leq 48$. Checking all remaining cases $2 \leq x \leq 48$ leads to the solutions $x \in \{14, 15, 47\}$. Of course, the number of cases to consider can be reduced by more refined case distinctions, e.g. with respect to $d(x)$. □