47^{th} Austrian Mathematical Olympiad

Regional Competition (Qualifying Round)

March 31, 2016

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1. Determine all positive integers k and n satisfying the equation
$$k^2 - 2016 = 3^n.$$
(Stephan Wagner)

2. Let a, b, c and d be real numbers with $a^2 + b^2 + c^2 + d^2 = 4$.
Prove that the inequality
$$(a + 2)(b + 2) \geq cd$$
holds and give four numbers a, b, c and d such that equality holds.
(Walther Janous)

3. On the occasion of the 47^{th} Mathematical Olympiad 2016 the numbers 47 and 2016 are written on the blackboard. Alice and Bob play the following game: Alice begins and in turns they choose two numbers a and b with $a > b$ written on the blackboard, whose difference $a - b$ is not yet written on the blackboard and write this difference additionally on the board. The game ends when no further move is possible. The winner is the player who made the last move.
Prove that Bob wins, no matter how they play.
(Richard Henner)

4. Let ABC be a triangle with $AC > AB$ and circumcenter O. The tangents to the circumcircle at A and B intersect at T. The perpendicular bisector of the side BC intersects side AC at S.
   (a) Prove that the points A, B, O, S and T lie on a common circle.
   (b) Prove that the line ST is parallel to the side BC.

   (Karl Czakler)

Working time: 4 hours.
Each problem is worth 8 points.