# $57^{\text{th}}$ Austrian Mathematical Olympiad

National Competition—Final Round—Solutions

27th/28th May 2026

**Problem 1.** *Let $ABC$ be an acute triangle with orthocenter $H$ and circumcircle $k$. Let $S$ be the intersection point of the tangents to $k$ at points $A$ and $B$, let $M$ be the midpoint of $AB$, and let $H'$ be the reflection of $H$ with respect to $AB$. Let $P$ be the intersection point of the line $MH$ with the circumcircle $k$ such that $H$ lies between $M$ and $P$.*

*Prove that the points $S$, $H'$ and $P$ are collinear.*

*(Karl Czakler)*

*Solution.* We consider three circles, namely the circumcircles of $ABC$, $AMO$ and $OMP$.

First of all, we note that the tangent of the circumcircle of $AMO$ in $A$ is also the tangent of the circumcircle of $ABC$ in $A$, since the homothety with center $A$ and factor $\frac{1}{2}$ maps the circumcircle of $ABC$ onto the circumcircle of $AMO$.

Furthermore, we note that the point $H'$ must lie on the circumcircle of $OMP$. This follows from consideration of the point $Q$, which is both symmetric to $H'$ with respect to $OM$ and symmetric to $C$ with respect to $O$. In the isosceles triangle $OQP$, we have $\angle OPQ = \angle PQO$, and due to the symmetry with respect to $OM$, we have $\angle MQO = \angle OH'M$. This yields

$$\angle OH'M = \angle MQO = \angle PQO = \angle OPQ = \angle OPM,$$

which shows that the quadrilateral $OMH'P$ is cyclic.

We now consider the radical axes of each pair of these circles.

The radical axis of the circumcircles of $AMO$ and $ABC$ is their common tangent in $A$. The radical axis of the circumcircles of $ABC$ and $OMH'P$ is the line $PH'$. Finally, the radical axis of the circumcircles of $OMH'P$ and $AMO$ is the line $OM$. These three lines therefore meet in the common radical center $S$ of the three circles, and by reasons of symmetry, this point also lies on the tangent of the circumcircle of $ABC$ in $B$.

*(Georg Weisbier)* □

**Problem 2.** *Alice and Bob play a game. In the beginning, there is a pile of $n$ stones on the table, where $n$ is a positive integer. In each move, either one stone is removed from an arbitrary pile, or one pile is split into two non-empty piles. The two players alternate, and Alice begins. When a player has no valid moves, the other player wins.*

*Determine for each $n$ which of the two players has a winning strategy.*

*(Theresia Eisenkölbl)*

*Solution.* For $n = 1$, Alice can remove the stone and has won.

For even $n$, Alice can divide the pile into two piles of $n/2$ stones. Afterwards, she can imitate each move of Bob in one half of the stones in the other half of the stones and returns the stones to a position where each pile occurs twice. Since piles can be divided at most $n - 1$ times, the number of stones has to decrease and the game always ends. Alice wins since she is always able to make a move.

For odd $n \geq 3$, we will prove by induction that Bob has a winning strategy.

For $n = 3$, Alice has two possibilites. She can either remove a stone or divide the pile into two piles of sizes 1 and 2. In the first case, Bob divides the pile into two piles of one stone each, in the second case, he removes one stone from the larger stone, and again leaves Alice two piles of one stone each. Afterwards, both remove one stone and Bob has won.

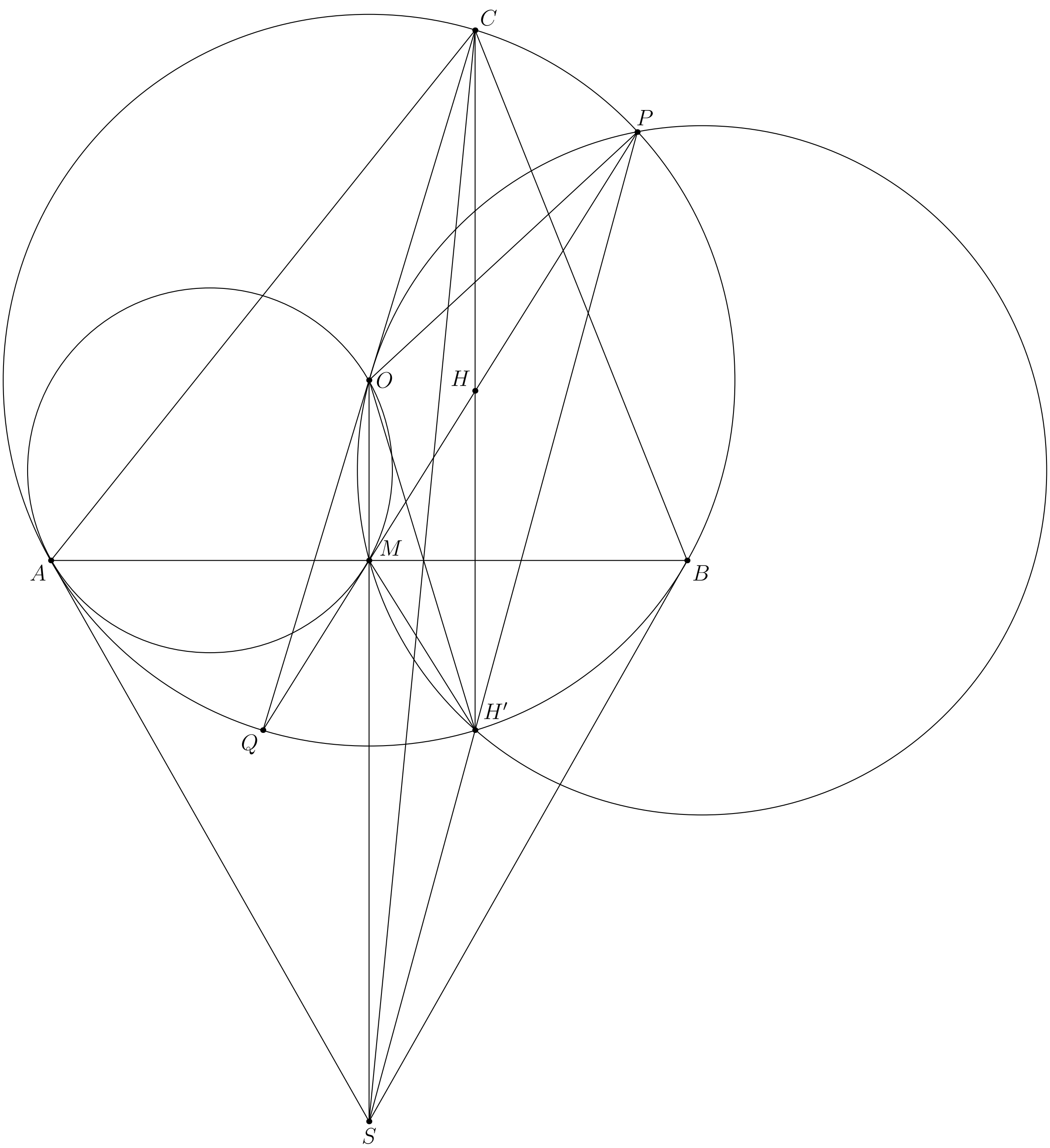


Figure 1: Problem 1

Consider now odd $n > 3$. We assume that Bob already has a winning strategy for odd $1 < k < n$.

If Alice removes a stone, then Bob can use the previously described winning strategy for Alice for an even number of stones and wins.

If Alice divides the pile so that one of the two piles has size 1, then Bob divides the the pile of size $n-1$ into another pile of size 1 and a pile of size $n-2$. If at any point in the game, Alice removes one of the two piles of size 1, Bob removes the other one. If at any point in the game Alice plays a move with the stones coming from the pile of size $n-2$, Bob will follow his winning strategy for $n-2$ which exists by induction hypothesis.

If Alice divides the starting pile so that no new pile has size 1, then Bob has a pile of even size and a pile of odd size $k$ with $3 \leq k < n$. Bob now divides the even pile into two piles of equal size. If at any point in the game Alice plays with the stones coming from the pile of size $k$, then Bob will use his winning strategy for $k$ stones which exists by induction hypothesis. If at any point in the game Alice plays a move with the stones coming from the two piles of equal size, then Bob answers symmetrically with the stones coming from the other of these two piles. Since Bob always has a move, Alice loses.

*(Theresia Eisenkölbl)* □

**Problem 3.** *Determine all values $a$ and $b$ such that there exists a sequence $(u_n)_{n\geq 1}$ of positive integers with $u_1 = a$, $u_2 = b$, and*

$$u_{2n} = \mathrm{GM}(u_{2n+1}, u_{2n-1}) \quad \textit{and} \quad u_{2n+1} = \mathrm{AM}(u_{2n}, u_{2n+2}) \quad \textit{for } n \geq 1.$$

*Here, $\mathrm{AM}(x, y)$ denotes the arithmetic and $\mathrm{GM}(x, y)$ the geometric mean of the two numbers $x$ and $y$.*

*(Theresia Eisenkölbl)*

*Solution.* The numbers $a$ and $b$ have to be positive integers. Furthermore, we have for $a > b$, that the sequence of positive integers is strictly decreasing which is impossible. Therefore, we have $a \leq b$. Since $u_3 = \frac{b^2}{a}$, we must have $a \mid b^2$.

Now, we must prove that under these conditions, we get a sequence of positive integers. Since $a \leq b$, the sequence is increasing. It is now enough to check that $u_3$ and $u_4$ are integers and that $u_3 \mid u_4^2$, since $0 < u_3 \leq u_4$ is true automatically from the monotonicity of the sequence. Under these conditions, we can repeat the argument and get only positive integers by induction.

Since we assume that $a \mid b^2$, we have that $u_3$ is an integer. Furthermore, $u_4 = 2u_3 - u_2$ is also clearly an integer.

Now, it is easy to see that $\dfrac{u_4^2}{u_3} = \frac{(2u_3-u_2)^2}{u_3} = (4u_3 - 4u_2) + \frac{u_2^2}{u_3} = 4u_3 - 4u_2 + u_1$ is an integer which finishes the proof.

*(Theresia Eisenkölbl)* □

**Problem 4.** *Determine all real numbers $\alpha$ for which a function $f : \mathbb{R} \to \mathbb{R}$ exists such that*

$$f(f(x) + y) = \alpha x + f(x + f(f(y)))$$

*for all real numbers $x$ and $y$.*

*(Walther Janous)*

*Answer.* For $\alpha = 0$ there are uncountably many solution functions $f$ and for $\alpha = 2$ there is exactly the solution $f(x) = -x$, $x \in \mathbb{R}$.

*Solution.* We denote our functional equation by $(FEQ)$.

For $\alpha = 0$, every constant function $f(t) = C$, $t \in \mathbb{R}$, with $C \in \mathbb{R}$, is a solution of $(FEQ)$.

Therefore, let $\alpha \neq 0$ in the following.

With $y = -f(x)$ in $(FEQ)$, we obtain

$$f(0) - \alpha x = f(x + f(f(f(-x)))),\ x \in \mathbb{R}.$$

Because the left-hand function is surjective, so is $f$. Therefore, there exists $b \in \mathbb{R}$ such that $f(b) = 0$.

With $x = b$, $(FEQ)$ yields

$$f(y) = \alpha b + f(b + f(f(y))),\ y \in \mathbb{R}. \quad (1)$$

Because $f$ is surjective, for every $z \in \mathbb{R}$ there exists a $y \in \mathbb{R}$ such that $z = f(y)$. Therefore, (1) implies that

$$z = \alpha b + f(b + f(z)),\ z \in \mathbb{R},$$

and also that $f$ is injective. Indeed, we have

$f(u) = f(v) \implies f(b + f(u)) = f(b + f(v)) \implies u - \alpha b = v - \alpha b \implies u = v.$

Thus, $f$ is bijective.

With $x = 0$ in $(FEQ)$, it follows

$$f(f(0) + y) = f(f(f(y))),\ y \in \mathbb{R}.$$

Thus, because of the bijectivity

$$f(0) + y = f(f(y)),\ y \in \mathbb{R}. \quad (2)$$

In particular, $y = 0$ implies that $f(0) = f(f(0))$ and therefore $f(0) = 0$.

Finally, with $y = 0$ in $(FEQ)$, we obtain

$$f(f(x)) = \alpha x + f(x),\ x \in \mathbb{R}.$$

This and (2) lead to $x = \alpha x + f(x),\ x \in \mathbb{R}$.

Therefore, any solutions of $(FEQ)$ must have the form

$$f(x) = (1 - \alpha)x,\ x \in \mathbb{R}.$$

All the obtained properties are necessary for $f$. Consequently, we must perform the check. We obtain for $(FEQ)$:

$$(1 - \alpha)((1 - \alpha)x + y) = \alpha x + (1 - \alpha)x + (1 - \alpha)^3 y,\ x, y \in \mathbb{R}$$

This simplifies to

$$((1 - \alpha)^2 - 1)x = ((1 - \alpha)^2 - 1)(1 - \alpha)y,\ x, y \in \mathbb{R}.$$

Consequently, $\alpha^2 - 2\alpha = 0$, that is, $\alpha \in \{0, 2\}$ must hold. (Otherwise, we would obtain that $x = (1 - \alpha)y$ would have to hold for all $x, y \in \mathbb{R}$, an evident contradiction.) Summary. The functional equation $(FEQ)$ has solutions for $\alpha \in \{0, 2\}$, namely:

- uncountably many for $\alpha = 0$ and
- exactly one for $\alpha = 2$, namely $f(x) = -x,\ x \in \mathbb{R}$.

*(Walther Janous)* □

**Problem 5.** *Let $ABCDEF$ be a convex hexagon with the property that each of the three segments joining the midpoints of opposite sides divides the total area of the hexagon in half.*

*Show that the three segments intersect in a common point.*

*(Walther Janous)*

*Solution.* Let $M_1$, $M_2$, $M_3$, $M_4$, $M_5$, $M_6$ be the midpoints of the segments $AB$, $BC$, $CD$, $DE$, $EF$ resp. $FA$. Let $X$ be the intersection of $M_2M_5$ and $M_3M_6$. We want to prove that $M_1$, $X$ and $M_4$ are collinear.

We denote the areas of the convex polygons $ABM_2XM_6$, $M_2CM_3X$, $M_3DEM_5X$ and $M_5FM_6X$ with $I$, $II$, $III$ resp. $IV$. The given property of the hexagon implies that $I + II = III + IV$ and $II + III = IV + I$. If we subtract the two equations, we get $I - III = III - I$, therefore $III = I$, which also implies $II = IV$.

Since $XM_6$ is a median in triangle $AXF$, we get $|AXM_6| = |M_6XF|$. Similarly, we get $|FXM_5| = |M_5XE|$ in triangle $FXE$, and therefore $|AXEF| = 2IV$. Analogously, we get $|BCDX| = 2II$, therefore $|BCDX| = |AXEF|$.

Again using the properties of medians, we get that $|AM_1X| = |M_1BX|$ and $|XDM_4| = |XM_4E|$. By appending adjacent triangles of equal areas to the polygons $BCDX$ and $AXEF$. we get $|M_4EFAM_1X| = |M_1BCDM_4X|$. Since the given property of the hexagon gives $|M_4EFAM_1| = |M_1BCDM_4|$, we conclude that the triangle $M_1XM_4$ has area 0 which implies the desired collinearity.

*(Walther Janous)* □

**Problem 6.** *On two strips, each consisting of* 100 *cells numbered from* 1 *to* 100*, there are* 50 *tokens each.*

*On the first strip, the tokens are located on cells* $1, 3, 5, \ldots, 99$. *Each token is then moved one or more cells to the right, i. e., to a cell with a higher number. The tokens may not jump over each other, and no cell may contain two or more tokens. Let $A$ be the number of possible configurations of the tokens that can be reached this way.*

*On the second strip, the tokens are located on cells* $1, 2, 3, \ldots, 50$. *Again, each token is moved one or more cells to the right. The tokens may not jump over each other, and no cell may contain two or more tokens. Moreover, a token cannot be moved further than to the position double its starting position. Let $B$ be the number of possible configurations for the tokens that can be reached this way.*

*Prove that $A = B$ holds.*

*(Stephan Wagner)*

*Solution.* For the first strip, let $a_i$ be the number of squares by which the $i$-th token is moved. The conditions are satisfied if $a_i \geq 1$ for all $i$, and if

$$a_1 \leq a_2 + 1 \leq a_3 + 2 \leq \cdots \leq a_{50} + 49, \tag{1}$$

so that the tokens cannot jump over each other or end up on the same square. The token at number 99 can only be moved one square. Therefore, $a_{50} = 1$, and it follows generally that $a_i \leq 51 - i$. The number $A$ is the number of ways to choose all $a_i$'s.

For the second strip, let $b_i$ be the number of squares by which the $i$-th token is moved. Again, $b_i \geq 1$ must hold, but additionally, $b_i \leq i$ by assumption. Furthermore,

$$b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_{50} \tag{2}$$

must hold so that the tokens cannot jump over each other or land on the same square. The number $B$ is the number of ways to choose all $b_i$'s.

If we set $b_i = i + 1 - a_{51-i}$, then the condition (1) becomes

$$51 - b_{50} \leq 50 - b_{49} + 1 \leq 49 - b_{48} + 2 \leq \cdots \leq 2 - b_1 + 49,$$

which is exactly equivalent to (2). Furthermore, $1 \leq a_i \leq 51 - i$ becomes $1 \leq b_i \leq i$. Therefore, this substitution represents a bijection between the respective choices, and it follows that $A = B$.

*(Stephan Wagner)* □