57th Austrian Mathematical Olympiad

National Competition—Preliminary Round—Solutions

2nd May 2026

Problem 1. *Prove that for all integers $n \geq 2$ the inequality*

$$\sqrt[2]{2+\sqrt[3]{3+\sqrt[4]{\ldots+\sqrt[n]{n}}}} < 2$$

holds.

(Walther Janous)

Solution.

- For $n = 2$, the inequality is clear because of $\sqrt{2} < 2$.
- Therefore, let $n \geq 3$ be given.
  We define the sequence $(w_k)_{n \geq k \geq 2}$ recursively as

  $$w_n = \sqrt[n]{n} \text{ and } w_k = \sqrt[k]{k + w_{k+1}} \quad \text{for } n-1 \geq k \geq 2$$

  and will proceed by induction from the inside out. For this, we need the auxiliary inequality valid for $k \geq 3$:

  $$\sqrt[k]{k+2} < 2. \quad (H)$$

  It is equivalent to $2^k > k+2$, i.e., $(1+1)^k > k+2$. However, using the binomial theorem, we have

  $$(1+1)^k > 1 + k + \frac{k(k-1)}{2} > 1 + k + \frac{2 \cdot 1}{2} = k + 2.$$

  We now show that for $n \geq k \geq 2$, the inequality $w_k < 2$ is satisfied.
  - For $k = n$, (H) yields that $w_n = \sqrt[n]{n} < \sqrt[n]{n+2} < 2$.
  - Let the inequality $w_k < 2$ hold for $k = n, n-1, \ldots, K$, with $K \geq 4$.
  - From this, using (H), it follows that

    $$w_{K-1} = \sqrt[K-1]{K - 1 + w_K} < \sqrt[K-1]{K - 1 + 2} = \sqrt[K-1]{K+1} < 2.$$

  - The estimate

    $$w_2 = \sqrt{2 + w_3} < \sqrt{2+2} = 2$$

    completes the proof of our inequality.

(Walther Janous) ☐

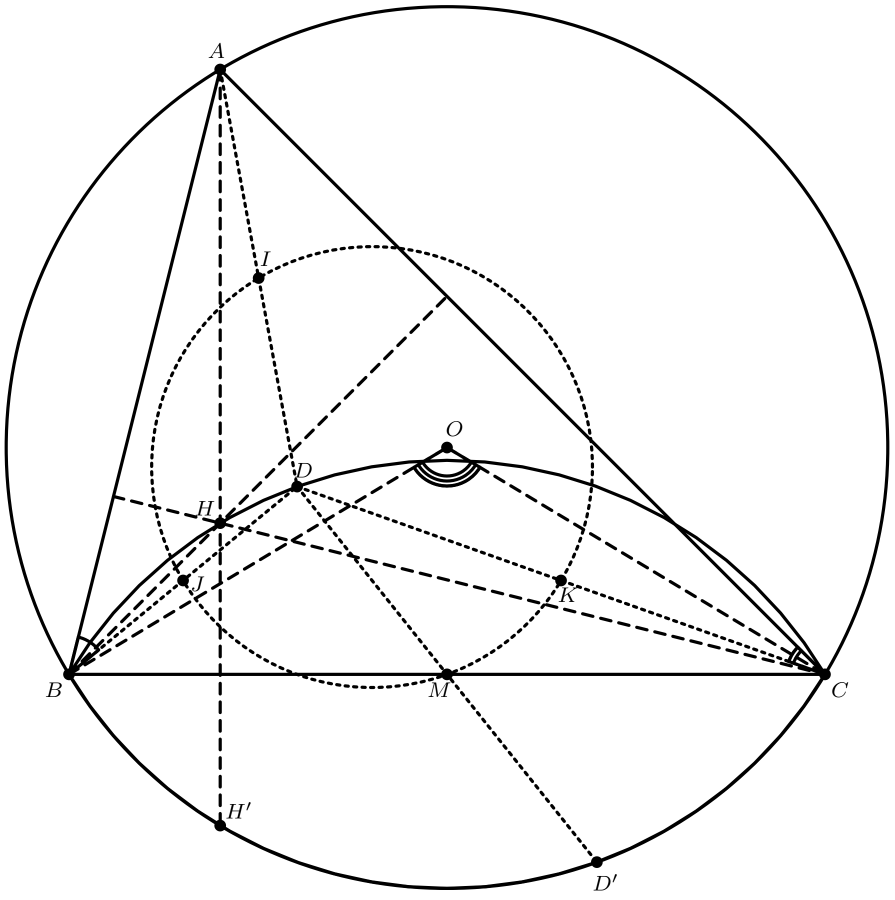
Problem 2. *Let ABC be an acute triangle. Let D be a point in its interior such that*

$$\angle DBA + 2 \cdot \angle BAC + \angle ACD = 180^\circ.$$

Let I, J and K denote the midpoints of segments DA, DB and DC, respectively.

Show that there exists a point X, independent of the choice of D, which lies on the circumcircle of triangle IJK.

(Dominik Pultar)



Solution. First, we shall determine the locus of all possible points D. To this end, we consider the circumcenter O of triangle ABC, which is suggested by the fact that $\angle BOC = 2\angle BAC$. Observe that $\angle CBO + \angle OCB + 2\angle BAC = 180°$. Thus, one possible choice for D is the point D_1 for which $\angle D_1BA = \angle CBO$ and $\angle ACD_1 = \angle OCB$, i.e. the isogonal conjugate of O. It is well-known that this isogonal conjugate is the orthocenter H of triangle ABC. Every other point D with $\angle DBA + \angle ACD + 2\angle BAC = 180°$ satisfies $\angle DBA + \angle ACD = \angle HBA + \angle ACH$, and hence

$$\angle DBH = \angle DBA - \angle HBA = \angle ACH - \angle ACD = \angle DCH.$$

Thus, D lies on the circle passing through B, C und H.

It is well-known that the mirror image of H across BC lies on the circumcircle k of triangle ABC. Conversely, D lies on the mirror image k' of k across BC. By symmetry, k' is also the image of k under point reflection across M. Hence, the image D' of D under this point reflection must lie on k. But this implies that M lies on the image of k under a homothety centered at D with scale factor $\frac{1}{2}$ – but this image is exactly the circle IJK, proving the claim (with $X = M$).

(Josef Greilhuber) □

Problem 3. *A list of numbers is called* productive *if it includes two distinct numbers whose product is included in the list.*

What is the smallest number k such that we can obtain a list that is not productive by removing k numbers from the list

$$1, 2, \ldots, 2025, 2026\,?$$

(Walther Janous)

Answer. 44

Solution. Since $45^2 = 2025$, we have $45 \cdot 46 > 2026$. We therefore note that we can remove the numbers $1, 2, 3, \ldots, 44$ from the list, after which the list

$$45, 46, 47, \ldots, 2025, 2026$$

remains. This remaining list is certainly non-productive, and we see that $k \leq 44$ certainly holds.

We must now determine whether $k < 44$ can hold. This is not possible. In order to see why this is the case, we consider the 43 triples

$$(2, 87, 2 \cdot 87), (3, 86, 3 \cdot 86), \ldots, (44, 45, 44 \cdot 45).$$

All of the numbers in these triples are present in the original list, and no number occurs twice. We see that one number must be removed from each of these triples in order for the remaining list to be non-productive. Furthermore, since $1 \cdot a = a$ holds for all a, the number 1 must also be removed from the list. We see that at least 44 numbers must be removed from the list in order to obtain a non-productive list, and we therefore obtain $k = 44$.

(Walther Janous) □

Problem 4. *A sequence $(a_n)_{n \geq 0}$ of positive integers is called k-*beautiful *if, for all $n \geq 0$, the property $a_n = a_{n+k}$ and, for all $n \geq 1$, the property*

$$a_{n-1} + a_{n+1} \mid a_n^2 + k$$

is satisfied. For which positive integers k is there a k-beautiful sequence?

(Dominik Pultar, Jan Strehn)

Answer. For all k, except for $k = 2$.

Solution. For all odd values of k, the constant sequence given by $a_n = 1$ for all n is obviously k-beautiful, since $a_n = a_{n+k} = 1$ for all $n \geq 0$, and $2 = a_{n-1} + a_{n+1}$ is a divisor of $a_n^2 + k = 1 + k$.

For even k, there is also always a k-beautiful sequence, provided that $k \geq 4$. To show this, we first observe that a_{n-1}, a_n, a_{n+1} satisfy the second condition in each of the following cases:

- $(11k)$: $a_{n-1} = a_n = 1$, $a_{n+1} = k$, as $a_{n-1} + a_{n+1} = 1 + k$ divides $a_n^2 + k = 1 + k$.
- $(1k1)$: $a_{n-1} = a_{n+1} = 1$, $a_n = k$, as $a_{n-1} + a_{n+1} = 2$ divides $a_n^2 + k = k^2 + k$.
- $(k11)$: $a_n = a_{n+1} = 1$, $a_{n-1} = k$, as $a_{n-1} + a_{n+1} = 1 + k$ divides $a_n^2 + k = 1 + k$.
- $(kk1)$: $a_{n-1} = a_n = k$, $a_{n+1} = 1$, as $a_{n-1} + a_{n+1} = 1 + k$ divides $a_n^2 + k = k^2 + k$.
- $(1kk)$: $a_n = a_{n+1} = k$, $a_{n-1} = 1$, as $a_{n-1} + a_{n+1} = 1 + k$ divides $a_n^2 + k = k^2 + k$.

With this observation, we can now construct k-beautiful sequences for all even $k \geq 4$, distinguishing three cases:

- $k \equiv 0 \bmod 3$: here we use the sequence $1, 1, k, 1, 1, k, \ldots$, which has a period of 3 and thus also satisfies the condition $a_n = a_{n+k}$. The three values a_{n-1}, a_n, a_{n+1} always fall into one of the cases above.
- $k \equiv 1 \bmod 3$: here we use the sequence that begins with the k values

  $$1, 1, k, 1, 1, k, \ldots, 1, 1, k, 1, 1, k, k$$

  ($\frac{k-4}{3}$ times the block $1, 1, k$, followed by $1, 1, k, k$) and then continues periodically (i.e., the block of the first k values is repeated). Thus, the condition $a_n = a_{n+k}$ is satisfied, and the three values a_{n-1}, a_n, a_{n+1} always fall into one of the cases above.

- $k \equiv 2 \bmod 3$: here we use the sequence that begins with the k values

  $$1, 1, k, 1, 1, k, \dots, 1, 1, k, 1, 1, k, k, 1, 1, k, k$$

  ($\frac{k-8}{3}$ times the block $1, 1, k$, followed by twice the block $1, 1, k, k$) and then continues periodically. As in the previous case, the condition $a_n = a_{n+k}$ is satisfied, and the three values a_{n-1}, a_n, a_{n+1} always fall into one of the cases above.

It remains to show that there are no 2-beautiful sequences. Suppose $(a_n)_{n\geq 0}$ were such a sequence. Let $a = a_0 = a_2 = \cdots$ and $b = a_1 = a_3 = \cdots$. Then $2a = a_0 + a_2$ must be a divisor of $b^2 + 2 = a_1^2 + 2$, and likewise $2b = a_1 + a_3$ must be a divisor of $a^2 + 2 = a_2^2 + 2$. In particular, $a^2 + 2$ and $b^2 + 2$ are even, and therefore a and b are also even. However, we now have $a^2 + 2 \equiv b^2 + 2 \equiv 2 \bmod 4$, so $a^2 + 2$ and $b^2 + 2$ are not divisible by 4. On the other hand, $2a$ and $2b$ are multiples of 4, so we arrive at a contradiction.

(Stephan Wagner) ☐