# $56^{\text{th}}$ Austrian Mathematical Olympiad

## Junior Regional Competition—Solutions

12th June 2025

---

**Problem 1.** *Determine all triples $(a, b, c)$ where $a$, $b$, $c \geq 0$ are integers which satisfy*

$$ab + bc + ca = 2(a + b + c).$$

*(Karl Czakler)*

*Answer.* The 17 solutions are all the permutations of the triples $(0, 0, 0)$, $(0, 3, 6)$, $(0, 4, 4)$, $(1, 2, 4)$, $(2, 2, 2)$.

*Solution.* As the problem is symmetric in $a$, $b$ and $c$, we can without loss of generality assume $a \leq b \leq c$. We can write the given equation as

$$a(b - 2) + b(c - 2) + c(a - 2) = 0.$$

This implies $a \leq 2$. We now consider three cases:

- $a = 2$
  Then we get $2b - 4 + bc - 2b = 0$, so $bc = 4$ and therefore $(b, c) = (1, 4)$ or $(b, c) = (2, 2)$. Only the latter option satisfies our assumption. So in this case, $(2, 2, 2)$ is the only solution.

- $a = 1$
  Then we get $b + bc + c = 2 + 2b + 2c$ and thus $(b - 1)(c - 1) = 3$. The case $b - 1 = -3$ and $c - 1 = -1$ is not possible, since $b$ is required to be positive. We therefore have $b - 1 = 1$ and $c - 1 = 3$. Hence, the only solution in this case is $(1, 2, 4)$.

- $a = 0$
  This implies $bc = 2b + 2c$ and thus $(b - 2)(c - 2) = 4$. The only possibilities are:
  - $b - 2 = 1$ and $c - 2 = 4$. We obtain the solution $(0, 3, 6)$.
  - $b - 2 = -4$ and $c - 2 = -1$. In this case, $b$ is negative, so we obtain no solution.
  - $b - 2 = 2$ and $c - 2 = 2$. We obtain the solution $(0, 4, 4)$.
  - $b - 2 = -2$ and $c - 2 = -2$. We obtain the solution $(0, 0, 0)$.

Therefore, the solutions are the triples $(2, 2, 2)$, $(1, 2, 4)$, $(0, 3, 6)$, $(0, 4, 4)$ and $(0, 0, 0)$, along with all of their permutations.

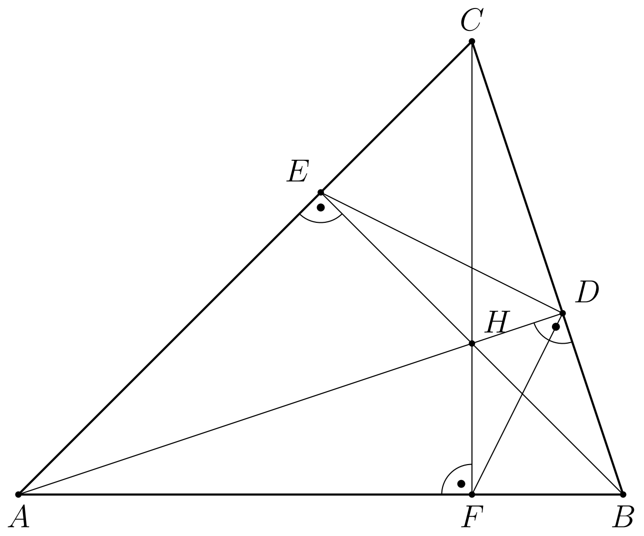
*(Karl Czakler)* □

**Problem 2.** *Let $ABC$ be an acute-angled triangle. Let the points $D, E$ and $F$ be the feet of the altitudes through the vertices $A, B$ and $C$.*

*Show: If $AF = CF$, then the segments $DE$ and $DF$ are perpendicular.*

*(Karl Czakler)*

*Solution.* The triangle $AFC$ is isosceles and right-angled, so we have $\angle FAC = \angle ACF = 45°$. Since $HECD$ is cyclic, we have $\angle EDH = \angle ECH = \angle ACF = 45°$.




The triangle $AEB$ is also right-angled, so we have $\angle EBA = 90° - \angle BAE = 90° - \angle FAC = 45°$. The quadrilateral $HFBD$ is cyclic, so we have $\angle HDF = \angle HBF = \angle EBA = 45°$.

This gives $\angle EDF = \angle EDH + \angle HDF = 90°$ as desired.

*(Karl Czakler)* □

**Problem 3.** *How many four-digit-numbers are there in the decimal system such that the thousands digit is greater than all other digits?*

*(Gerhard Kirchner)*

*Answer.* 2025

*Solution.* We denote the thousands digits by $k \in \{1, \dots, 9\}$. Each of the other digits can be chosen from the set $\{0, \dots, k-1\}$, we therefore have $k$ possible choices for each digit and $k^3$ possibilities for the block consisting of the 3 last digits. In total we have

$$1^3 + 2^3 + \ldots + 9^3 = \left(\frac{9 \cdot 10}{2}\right)^2 = 45^2 = 2025$$

possibilities by a well-known formula. □

**Problem 4.** *Consider all numbers of the form*

$$20^n + 2 \cdot 5^n,$$

*where $n \geq 2025$ is an integer.*

*Determine the largest integer $k$ that divides each of these numbers.*

*(Walther Janous)*

*Answer.* $k = 6 \cdot 5^{2025}$

*Solution.* We have $20^n + 2 \cdot 5^n = 5^n(4^n + 2) = 5^{2025} \cdot 5^{n-2025} \cdot 2 \cdot (2^{2n-1} + 1)$. Because of $n - 2025 \geq 0$, we get $k = 5^{2025} \cdot 2 \cdot d$, where $d$ is the greatest common divisor of all numbers of the form $2^{2n-1} + 1$ for $n \geq 2025$.
But we also have

$$\gcd(2^{2n-1} + 1, 2^{2n+1} + 1) = \gcd(2^{2n-1} + 1, 2^{2n+1} + 1 - 4(2^{2n-1} + 1)) = \gcd(2^{2n-1} + 1, -3) \in \{1, 3\}.$$

Since $2^{2n-1} + 1 \equiv (-1)^{2n-1} + 1 \equiv 0 \pmod 3$, we conclude that $d = 3$ and therefore $k = 6 \cdot 5^{2025}$.

*(Walther Janous)* □