56^{th} Austrian Mathematical Olympiad

National Competition—Final Round—Solutions

28th/29th May 2025

Problem 1. *Let k and n be positive integers.*
Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$x^k f(y) - y^n f(x) = f\left(\frac{y}{x}\right)$$

for all real numbers x and y with $x \neq 0$.

(Walther Janous)

Answer.

- $k \neq n$: $f(x) = 0$, $x \in \mathbb{R}$
- $k = n$:
$$f(x) = \begin{cases} C(x^k - x^{-k}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$
  where $C \in \mathbb{R}$.

Solution. We denote the functional equation by (F).

- $y = 0$ in (F) leads to $x^k f(0) = f(0)$ for all $x \neq 0$, whence $f(0) = 0$.
- $y = x$ (where $x \neq 0$) in (F) yields
$$(x^k - x^n) f(x) = f(1). \qquad (1)$$
  In particular we get for $x = 1$: $0 \cdot f(1) = f(1)$, so $f(1) = 0$.
- $y = 1$ in (F) results in $0 - f(x) = f\left(\frac{1}{x}\right)$. This means, we have for all $x \neq 0$:
$$f\left(\frac{1}{x}\right) = -f(x). \qquad (2)$$
  In particular we get for $x = -1$: $f(-1) = -f(-1)$, so $f(-1) = 0$.
- In summary, we have shown $f(x) = 0$ for $x \in \{-1, 0, 1\}$.

We will now distinguish between two cases.

(i) $k \neq n$. Using (1) implies $f(x) = 0$ for $x^k - x^n \neq 0$. But all real solutions of $x^k - x^n = 0$ are elements of $\{-1, 0, 1\}$. Therefore, $f(x) = 0$ is the unique solution, This can easily be confirmed by checking..

(ii) $k = n$. We replace x ($\neq 0$) by $\frac{1}{x}$ in (F) and get
$$x^{-k} f(y) - y^k f\left(\frac{1}{x}\right) = f(xy),$$
that is, because of (2),
$$x^{-k} f(y) + y^k f(x) = f(xy). \qquad (3)$$
Swapping x and y leads to
$$y^{-k} f(x) + x^k f(y) = f(xy).$$

This and (3) imply

$$(y^k - y^{-k})f(x) = (x^k - x^{-k})f(y)$$

for all $x, y \neq 0$. Thus we have for all $x, y \neq 0$ subject to $x^k - x^{-k} \neq 0$ and $y^k - y^{-k} \neq 0$:

$$\frac{f(x)}{x^k - x^{-k}} = \frac{f(y)}{y^k - y^{-k}}.$$

Therefore, there exists a real constant C such that

$$f(x) = C(x^k - x^{-k})$$

for $x \neq 0$ and $x^k - x^{-k} \neq 0$, that is $x \notin \{-1, 0, 1\}$. Since $f(-1) = 0$ and $f(1) = 0$, the function term also gives the correct values for $x \in \{-1, 1\}$. Consequently, all functions are of the form

$$f(x) = \begin{cases} C(x^k - x^{-k}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

where $C \in \mathbb{R}$. We check that they are solution as follows.

- For $x, y \neq 0$ (F) becomes

$$Cx^k(y^k - y^{-k}) - Cy^k(x^k - x^{-k}) = C(y^k x^{-k} - x^k y^{-k}) = C\left(\left(\frac{y}{x}\right)^k - \left(\frac{y}{x}\right)^{-k}\right),$$

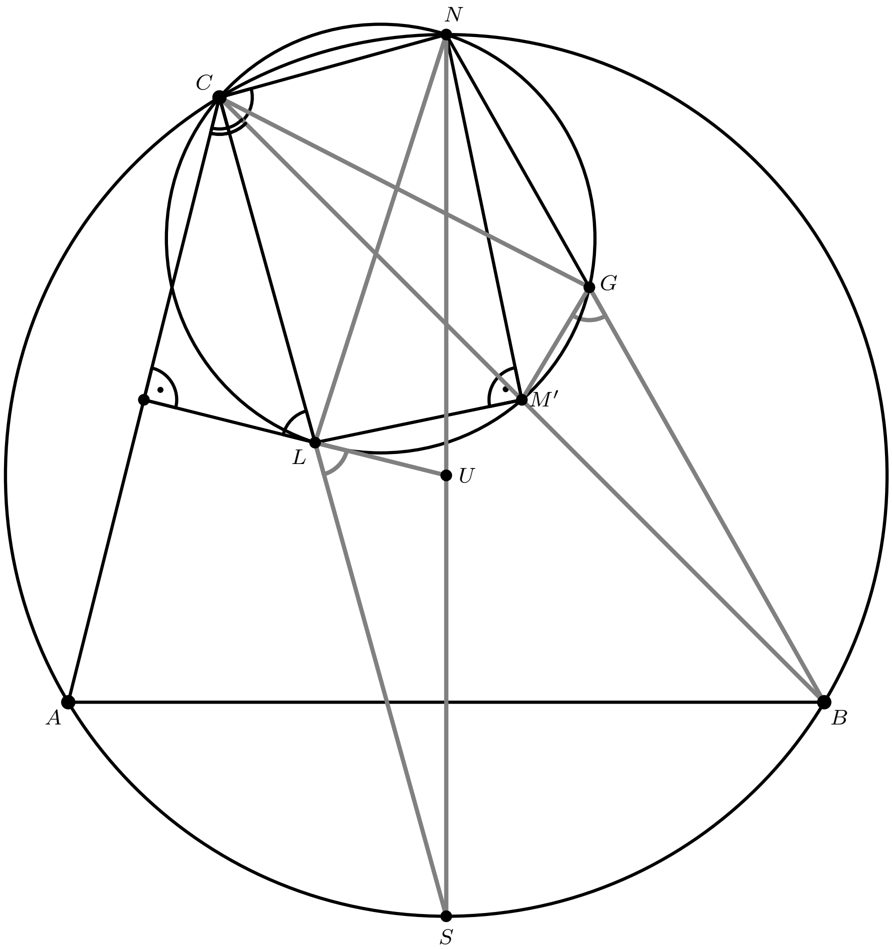
- while for $y = 0$, the equation (F) becomes $0 - 0 = 0$.

(Walther Janous) □

Problem 2. *Let ABC be a triangle with $AC < BC$. Let L be the point of intersection of the angle bisector of $\angle ACB$ with the perpendicular bisector of AC. Let M be the midpoint of segment BC and let N be the midpoint of the arc from A to B of the circumcircle which contains C.*

Show that LMN is a right triangle.

(Karl Czakler)



Solution.

Let U be the circumcenter of the triangle ABC and let S be the midpoint of the arc AB on the circumcircle that does not contain C. Let G and M' be the second points of intersection of the circumcircle of LNC with BN and BC, respectively. We want to prove that $M' = M$.

Since S and N are south pole and north pole of the triangle ABC, the angle bisector of $\angle BCA$ is perpendicular to CN and passes through S. By the inscribed angle theorem, we get

$$\angle GBC = \angle NBC = \angle NSC = \angle NSL$$

and

$$\angle CGB = 180^\circ - \angle NGC = 180^\circ - \angle NLC = \angle SLN.$$

Therefore, the triangles BGC and SLN are similar.

Again, by the inscribed angle theorem, we get

$$\angle M'GB = \angle M'CN = \angle BCN$$

and since NC is perpendicular to CS, LU is perpendicular to AC and CS is the angle bisector of $\angle ACB$, we get

$$\angle M'GB = \angle BCN = 90^\circ - \angle SCB = 90^\circ - \angle ACL = \angle SLU.$$

Since the triangles BGC and SNL are similar, U is the midpoint of SN and $\angle M'GB = \angle SLU$, we get that M' is the midpoint of the segment BC and therefore, identical to M. So, C, L, M and N lie on a circle, and thus $\angle LCN = 90^\circ$ implies $\angle NML = 90^\circ$.

(Karl Czakler, Josef Greilhuber) □

Problem 3. *Anna and Bertha play two games on a regular 2025-gon. One of these is the Triangle Game and the other is the Quadrilateral Game. In both games, a move consists of drawing a diagonal of the 2025-gon that has not been drawn previously and does not intersect any previously drawn diagonal in an inner point. The players take alternate moves starting with Anna. The game ends when no additional allowed diagonal can be drawn.*

(a) The Triangle Game: If one or two triangles are created when the diagonal is drawn, the player labels the resulting triangle(s) with her initial. The player with the most labeled triangles at the end of the game wins. If they have the same number of labeled triangles, the game ends in a tie. Does Anna have a winning strategy, does Bertha have a winning strategy, or must the game end in a tie if both players play in an optimal way?

(b) The Quadrilateral Game: If one or two quadrilaterals with no diagonals inside are created when the diagonal is drawn, the player labels the resulting quadrilateral(s) with her initial. No diagonals may be drawn in labeled quadrilaterals. The player with the most labeled quadrilaterals at the end of the game wins. If they have the same number of labeled quadrilaterals, the game ends in a tie. Does Anna have a winning strategy, does Bertha have a winning strategy, or must the game end in a tie if both players play in an optimal way?

(Theresia Eisenkölbl)

Answer. 1. Answer: Bertha has a winning strategy.
2. Answer: Anna has a winning strategy.

Solution. (a) We will prove by induction that Bertha has a winning strategy for any convex polygon with an odd number $n > 3$ of sides.

For $n = 5$, this is obviously true, since for any diagonal chosen by Anna, any diagonal chosen by Bertha gives an advantage of 1 point.

For general odd n, Bertha can assume without loss of generality, that the polygon is regular. Bertha chooses one of the endpoints of Anna's first diagonal, reflects this diagonal with respect to the the symmetry axis of the polygon through the chosen endpoint, and draws this mirror image.

The two diagonals divide the polygon in three polygons. One of them has both diagonals as sides. If it is a triangle, then Bertha has an advantage of 1 point from this triangle. Otherwise, it is an odd polyon with $n > 3$ sides, and by induction hypothesis, Bertha can obtain an advantage of at least one point in this polygon as long as she always answers moves of Anna in this polygon. (This is possible since there will always be $k - 3$ diagonals in a convex k-gon that is fully divided into triangles, so there is an even number of moves inside this polygon.)

For the two outer polygons that each have just one of the two first diagonals as sides, they are either both triangles, giving the players one point each, or Bertha can simply mirror any move of Anna in one of them in the other one. This guarantees that Anna and Bertha get the same number of points in the outer polygons.

Bertha will win with an advantage of 1 point if she follows this strategy.

(b) We call a convex polygon of order r if it has $r + 2$ vertices (which means that it could be divided into r triangles by $r - 1$ non-intersecting diagonals). A diagonal divides the polygon into two polygons of orders $a, b \geq 1$ with $a + b = r$.

We will now determine for all $r \geq 3$ (a pentagon) the maximal difference in points that the first player can guarantee and write down this difference.

$n = 3$: $\boxed{1}$. Anna cuts off a quadrilateral ($a = 2$), and the game ends with 1 point advantage for Anna.

$n = 4$: $\boxed{2}$. Anna divides the polygon into two quadrilaterals. The game ends with 2 points advantage for Anna.

$n = 5$: $\boxed{0}$. The choice $a = 1$, $b = 4$ gives 2 points for Bertha. The choice $a = 2$, $b = 3$ gives 0 points difference. So Anna will choose the second option.

$n = 6$: $\boxed{0}$. The choice $(a, b) = (1, 5)$ gives 0 points difference, the choice $(2, 4)$ gives 1 point for Bertha. The choice $(3, 3)$ gives 0 points difference (both players take one quadrilateral). So Anna will choose $(1, 5)$ or $(3, 3)$.

$n = 7$: $\boxed{1}$. The choice $(1, 6)$ gives 0 points difference. The choice $(2, 5)$ gives 1 point advantage for Anna. The choice $(3, 4)$ gives 1 point advantage for Bertha (who will play in the polygon of order 4 to secure the 2 points there while Anna can take the 1 point in the polygon of order 3). So Anna will choose $(2, 5)$.

For $n > 7$, we have: If Anna cuts off a quadrilateral ($a = 2$), then she gets a point while Bertha gets the points for order $r - 2$. If Bertha gets 0 points advantage there, then Anna has 1 point advantage in total. If Berthat gets 1 point advantage there, then Anna has 0 points advantage in total.

If Anna chooses another division into two polygons, then Bertha can choose the better part for her move, but she can also answer Anna's moves in the other part. Anna can at most obtain 0 difference in points, but that is already guaranteed by the previous method which gives either 0 or 1. Note that it does not matter whether Bertha is forced at one point to play an extra move in a polygon because by induction, beginning in smaller polygons is never a disadvantage.

We get the following table for points advantage for the first player:

r	3	4	5	6	7	8	9	10	11	12	13	...
	1	2	0	0	1	1	0	0	1	1	0	...

This has a period of length 4. Since 2025 sides correspond to order 2023, we get that our r is congruent to 3 modulo 4, so the value at $r = 7$ is the one we seek. We conclude that Anna will have an advantage of 1 point if both sides play optimally.

(Theresia Eisenkölbl) □

Problem 4. *For a positive integer n, let $a_1, a_2, \ldots, a_n$ be positive real numbers with $a_1 a_2 \cdots a_n = 2^n$. Prove that*

$$a_1^2 + a_1 a_2^2 + a_1 a_2 a_3^2 + \ldots + a_1 a_2 \cdots a_{n-1} a_n^2 \geq 4(2^n - 1),$$

and determine when equality holds.

(Karl Czakler)

Solution. For all $k \geq 1$, we have $(a_k - 2)^2 \geq 0$, which can be rewritten as

$$a_k^2 \geq 4a_k - 4.$$

Equality only holds for $a_k = 2$. Thus it follows that

$$\begin{aligned} a_1^2 + a_1 a_2^2 + a_1 a_2 a_3^2 + \cdots + a_1 a_2 \cdots a_{n-1} a_n^2 &= \sum_{k=1}^{n} a_1 a_2 \cdots a_{k-1} a_k^2 \\ &\geq \sum_{k=1}^{n} a_1 a_2 \cdots a_{k-1}(4a_k - 4) \\ &= 4\sum_{k=1}^{n} a_1 a_2 \cdots a_k - 4\sum_{k=1}^{n} a_1 a_2 \cdots a_{k-1} \\ &= 4a_1 a_2 \cdots a_n - 4 = 4(2^n - 1). \end{aligned}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n = 2$.

(Karl Czakler) □

Problem 5. *Let ABC be a triangle. For every integer $n \geq 2$, point D_n lies on segment CB with $CD_n = \frac{1}{n}CB$, and point E_n lies on segment CA with $CE_n = \frac{1}{n+1}CA$.*

Prove that all lines D_nE_n pass through a common point.

(Walther Janous)

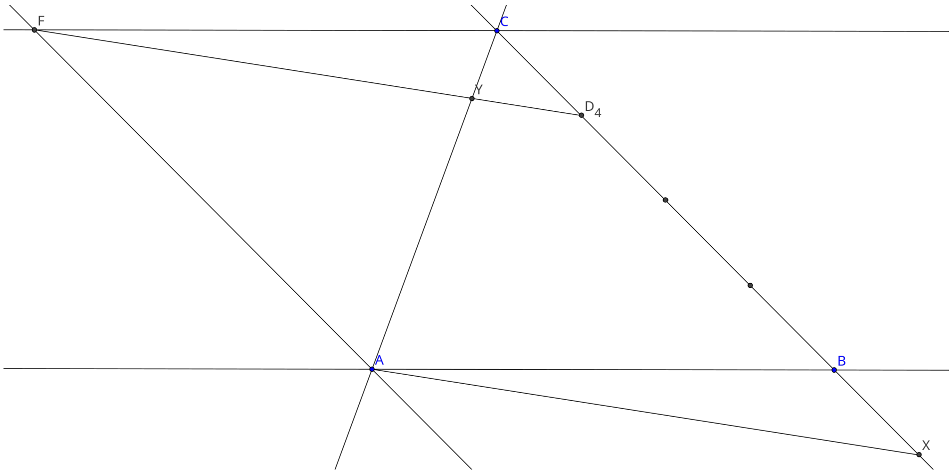


Figure 1: Problem 5

Solution. Let F be the point such that $ABCF$ is a parallelogram. We want to prove that F lies on D_nE_n.

Let X be the point on the extension of BC beyond B such that $CD_n = BX$. Because of symmetry, we have that FD_n is parallel to AX. Therefore, the intersection Y of FD_n and AC satisfies

$$CY : CA = CD_n : CX = \frac{CD_n}{CB + BX} = \frac{CD_n}{nCD_n + CD_n} = 1 : (n+1) = CE_n : CA.$$

This implies $Y = E_n$ which proves that F is on D_nE_n.

(Theresia Eisenkölbl) □

Problem 6. *Determine all positive integers n, such that there exist positive integers $z_1 < z_2 < \cdots < z_n$ satisfying* $\gcd(z_j, z_k) = z_k - z_j$ *for all j and k with $1 \le j < k \le n$.*

(Walther Janous)

Solution.

Lemma. Let $a < b$ be positive integers. Then $\gcd(a, b) = b - a$ holds if and only if $(b - a) \mid a$.

Proof. If $\gcd(a, b) = b - a$, then $(b - a) \mid a$ follows from the definition of the greatest common divisor.

Assume now that $(b - a) \mid a$ and set $d = b - a$. Then $a = dx$ for a suitable positive integer x by assumption and $b = a + (b - a) = dx + d = d(x + 1)$. Thus

$$\gcd(a, b) = \gcd(dx, d(x+1)) = d \gcd(x, x+1) = d,$$

as required. ■

A tuple $(z_1, \ldots, z_n)$ of positive integers is said to be a valid tuple if $\gcd(z_j, z_k) = z_k - z_j$ for all $1 \le j < k \le n$.

We now prove that a valid tuple exists for all $n \ge 1$ by induction on n. For $n = 1$, every 1-tuple is valid.

Assume now that $(z_1, \ldots, z_n)$ is a valid tuple. Set $A = \operatorname{lcm}(z_1, \ldots, z_n)$, $z_1' = A$ and $z_k' = A + z_{k-1}$ for $2 \le k \le n+1$. We claim that $(z_1', \ldots, z_k')$ is a valid tuple.

If $2 \le j < k \le n+1$, then

$$z_k' - z_j' = (A + z_{k-1}) - (A + z_{j-1}) = (z_{k-1} - z_{j-1}) \mid z_{j-1} \mid (z_{j-1} + A) = z_j' \qquad (1)$$

by the lemma, induction hypothesis and construction of A.

If $2 \leq k \leq n+1$, then

$$z'_k - z'_1 = (A + z_{k-1}) - A = z_{k-1} \mid A = z'_1 \tag{2}$$

by construction of A.

The lemma and (1) and (2) imply that $(z'_1, \ldots, z'_{n+1})$ is a valid tuple.

(Clemens Heuberger) □