CAPS Match 2025 - solutions

ISTA, Austria

(Second day – 18 June 2025)

4. The plane was divided by vertical and horizontal lines into unit squares. Determine whether it is possible to write integers into cells of this infinite grid so that:

(i) every cell contains exactly one integer

(ii) every integer appears exactly once

(iii) for every two cells A and B sharing exactly one vertex, if they contain integers a and b then at least one of the cells sharing a common side with both A and B contains an integer between a and b.

(*Marta Strzelecka and Michał Strzelecki, Poland*)

Solution. Yes, this is possible. Consider the spiral depicted below and write consecutive integers along the spiral:

-40	-39	-38	-37	-36	-35	-34	-33	-32
25	24	23	22	21	20	19	18	-31
26	-13	-12	-11	-10	-9	-8	17	-30
27	-14	5	4	3	2	-7	16	-29
28	-15	6	-1	0	1	-6	15	-28
29	-16	7	-2	-3	-4	-5	14	-27
30	-17	8	9	10	11	12	13	-26
31	-18	-19	-20	-21	-22	-23	-24	-25
32	33	34	35	36	37	38	39	40

We claim that this works. Consider any two cells A and B sharing exactly one vertex. Consider the 2×2 square containing A and B. If the 2×2 square contains a "corner" of the spiral then for some n and k the numbers in that 2×2 square are arranged in the following way (up to rotation or reflection):

$n+1$	k
n	$n-1$

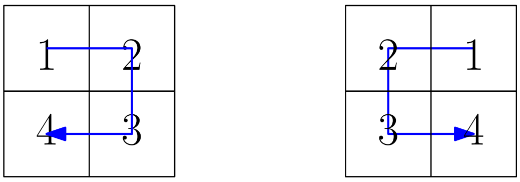
and therefore the conditions are satisfied no matter which opposite cells of the 2×2 square A and B are. Indeed, if A and B contain $n-1$ and $n+1$, then the good cell is the one containing n. If A and B contain n and k and $n < k$ then $n < n+1 < k$ and the good cell is the one containing $n+1$. If A and B contain n and k and $n > k$ then $k < n-1 < n$ and the good cell is the one containing $n-1$.

Otherwise, the numbers are arranged in the following way (again, up to rotation or reflection):

$k+1$	k
n	$n+1$

for some n, k, and again, the conditions are satisfied. Indeed, without loss of generality, assume $n < k$. Then $n < n+1 < k < k+1$. If A and B contain n and k then the good cell is the one containing $n+1$. Otherwise, A and B contain $n+1$ and $k+1$, and the good cell is the one containing k.

Alternatively, one can notice that condition (iii) from the problem statement means that we can orient each square 2×2 according to the increasing numbers as suggested in the picture below:

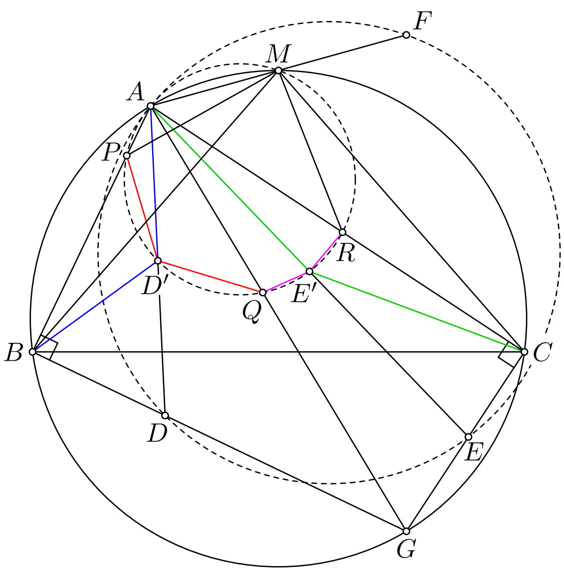


With this observation, it's relatively easy to check that the spiral construction satisfies this.

5. We are given an acute triangle ABC. Point D lies in the halfplane AB containing C and satisfies $DB \perp AB$ and $\angle ADB = 45° + \frac{1}{2}\angle ACB$. Similarly, E lies in the halfplane AC containing B and satisfies $AC \perp EC$ and $\angle AEC = 45° + \frac{1}{2}\angle ABC$. Let F be the reflection of A in the midpoint of arc BAC (containing point A). Prove that points A, D, E, F are concyclic. (*Patrik Bak, Slovakia*)

Solution 1. Denote $\angle ABC = \beta$ and $\angle ACB = \gamma$. The conditions translate as $\angle BAD = 45° - \gamma$ and $\angle EAC = 45° - \beta$. Denote by G the intersection point of BD and CE. Clearly $\angle BAG = 90° - \gamma = 2\angle BAD$, and so AD is the angle bisector of BAG. Similarly, AE is the angle bisector of GAC.

Let D', E' be the midpoints of AD, AE, respectively. It is enough to show that the circle through A, D', E' also passes through the midpoint of arc BAC. Consider the circumcircle of $AD'E'$ and denote its second interesection points with AB, AG, AC by P, Q, R, respectively.



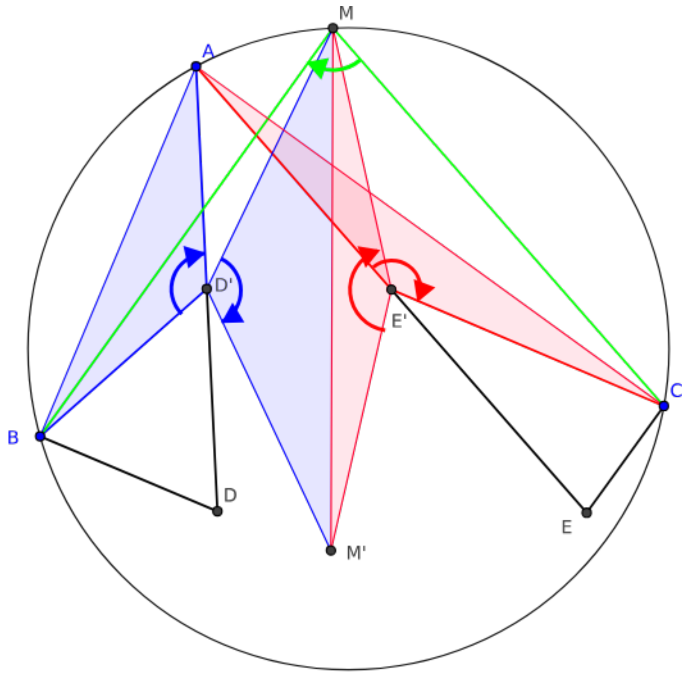
First, we will show that $BP = CR$. Notice that due $\angle ABD$ being right, we have that D' is the circumcenter of ABD, and so $D'B = D'A$. Then we get $\angle D'BA = \angle PAD' = \angle D'AQ$, and also $\angle AQD' = \angle BPD'$. Together with $D'B = D'A$, we have that triangles $D'AQ$ and $D'BP$ are congruent, and so $BP = AQ$. Similarly, we can show $CR = AQ$, and so $BP = CR$ as we wanted.

We will now show that the circle through A, P, R passes through the midpoint of arc BAC. Denote by M the second intersection of this circle with the circle ABC.

Clearly $\angle MBP = \angle MCR$ and $\angle MPA = \angle MRA$, and also $BP = RC$, so triangle MBP and MCR are congruent, giving $MB = MD$, which is enough.

Solution 2. Similarly to the previous solution, we consider homothety with center A and coefficient $1/2$ to obtain points D', E', M and prove that $AD'B, AE'C, BMC$ are isosceles triangles. Moreover, by angle chasing we can get $\angle BMC = \alpha$, $\angle AD'B = 90° + \gamma$ and $\angle AE'C = 90° + \beta$. Let us notice that the sum of these angles $\angle BMC + \angle AD'B + \angle AE'C = 360°$. We may view these three isosceles triangles as three rotations (for example, triangle $AD'B$ corresponds to the rotation around D' by angle $AD'B$ and sends point B to point A). We will call them green, blue, and red.

Because the sum of the three angles is $360°$, the composition of these three rotations is a translation. Moreover, if we follow the image of C we notice that green rotation maps it to B, then blue maps it to A, and finally red maps it back to C. Hence, the translation is actually an identity. Let M' be the image of M under the blue rotation. Then $MD'M'$ is similar to $AD'B$. And because M is the center of the green rotation, the composition of blue and red rotations has to map M back to M. Hence, $M'E'M$ has to be similar to $AE'C$. And so $\angle D'ME' = \angle BAD' + \angle CAE' = \alpha/2 = \angle D'AE'$. Thus, $AME'D'$ is cyclic and we are done.



6. Find all functions $f\colon (0,\infty) \to [0,\infty)$ such that for all $x, y \in (0,\infty)$ it holds that

$$f(x + yf(x)) = f(x)f(x + y).$$

(*Dominik Martin Rigász, Slovakia*)

Solution. Any f such that $f(x) \in \{0,1\}$ for all $x \in \mathbb{R}^+$works. Furthermore, any f such that

$$f(x) = \begin{cases} 0 \text{ or } 1 & x \in (0, x_0) \\ c & x = x_0 \\ 0 & x \in (x_0, \infty) \end{cases}$$

works as well, where $x_0 > 0, c \geq 0$ are arbitrary constants. We now show that these are the only solutions. For $f(x) \neq 0$ easy both-ways induction yields that for all $n \in \mathbb{Z}$ it is true that

$$f(x)^n f(x + y) = f\left(x + yf(x)^n\right) \tag{2}$$

Now assume there exist $0<x_0<x_1$ such that $f\left(x_0\right) \notin\{0,1\}$ and $f\left(x_1\right) \neq 0$ (if such a pair doesn't exist then f must have one of the two forms described above). Then substituting $\left[x_0, x_1-x_0\right]$ into (1) and manipulating n (in particular we consider $n \rightarrow-\infty$ if $f\left(x_0\right)<1$, and $n \rightarrow+\infty$ if $f\left(x_0\right)>1$) yields that f reaches arbitrarily large values at arbitrarily large arguments. Hence, for every pair of positive reals c_1, c_2 there are infinitely many x such that $x>c_1$ and $f(x)>c_2$. Call this fact $(\star)$.

We now multiply the given equation by $f(x+y+z)$, where z is a positive real number, to get

$$f(x+y+z) f(x+y f(x))=f(x) f(x+y) f(x+y+z)=f(x) f(x+y+z f(x+y)),$$

where we've used the property from the problem statement to obtain the second equality. We now choose z such that $z>y f(x)-y$. Then $x+y+z>x+y f(x)$. Hence, we can apply the problem statement on both the left-most side and the right-most side of the above equation to get

$$\begin{gathered}
f(x+y+z) f(x+y f(x))=f(x+y f(x)+(z-y f(x)+y) f(x+y f(x))) \\
f(x) f(x+y+z f(x+y))=f(x+(y+z f(x+y)) f(x))
\end{gathered}$$

Together with $f(x+y f(x))=f(x) f(x+y)$, since the LHS's are equal in the above two equations, we get

$$f(x+y f(x)+(z-y f(x)+y) f(x) f(x+y))=f(x+(y+z f(x+y)) f(x)). \quad (3)$$

If the arguments in the above equation were equal, then by simplification, this would yield the equivalent equality

$$(-y f(x)+y) f(x) f(x+y)=0. \quad (4)$$

We now choose x_0, y_0 such that $f\left(x_0\right) \notin\{0,1\}$ and $f\left(x_0+y_0\right) \neq 0$ and substitute $\left[x_0, y_0\right]$ into (2). Note that for this pair, equation (3) does not hold, and hence the arguments in (2) are always distinct. In particular, the arguments on both sides of (2) are linear functions in z with the same positive gradient (namely $f\left(x_0\right) f\left(x_0+y_0\right)$), but different y-intercept values. Since (2) holds for all large z (namely all $z>y_0 f\left(x_0\right)-y_0$), it follows that f is eventually periodic. Hence, there are constants $C, P>0$ (dependent on x_0, y_0), such that $f(x)=f(x+P)$ for all $x>C$.

By $(\star)$ we know that there is an $x_2>C$ such that $f\left(x_2\right) \notin\{0,1\}$. Then by comparing $\left[x_2, y\right]$ with $\left[x_2, y+P\right]$ in the original equation we get

$$f\left(x_2+y f\left(x_2\right)\right)=f\left(x_2+y f\left(x_2\right)+P f\left(x_2\right)\right),$$

since the RHS's remains the same (since $x_2+y>C$). Now let $y=\frac{P}{f\left(x_2\right)}$ in the above, to obtain

$$f\left(x_2+P\right)=f\left(x_2+P+P f\left(x_2\right)\right)$$

and hence $f\left(x_2\right)=f\left(x_2+P f\left(x_2\right)\right)=f\left(x_2\right) f\left(x_2+P\right)=f\left(x_2\right)^2$, clear contradiction.