CAPS Match 2025

ISTA, Austria

(First day – 17 June 2025)

1. Let a, b, c, d be nonnegative real numbers for which $a^2 + b^2 = ac + bd$ holds and c, d are not both zero. Find maximum and minimum value of the expression

$$\frac{ad + bc - cd}{c^2 + d^2}.$$

(*Michal Janík, Czech Republic*)

Solution 1. We will show that the maximum value is $\frac{1}{2}$ and the minimum is $-\frac{1}{2}$. For maximum, after some rearranging, we want to prove

$$2(ad + bc - cd) \leq c^2 + d^2,$$

or

$$2(ad + bc) \leq (c + d)^2.$$

After adding double the expression $ac + bd = a^2 + b^2$ to both sides of this inequality, we will get equivalent inequality

$$2(a + b)(c + d) = 2(ad + bc + ac + bd) \leq (c + d)^2 + 2a^2 + 2b^2,$$

which can be further rearranged to

$$0 \leq (c + d)^2 - 2(a + b)(c + d) + (a + b)^2 + (a - b)^2 \leq (c + d - a - b)^2 + (a - b)^2,$$

which clearly holds. Moreover, this maximal value is reached by $a = b = c = d > 0$.

For the minimum value, notice that if $a \geq c$ or $b \geq d$ holds, $ad + bc \geq ad \geq cd$ and the expression is non-negative. So for it to be negative, both $a < c, b < d$ must hold and $a^2 + b^2 < ac + bd$ if a and b wouldn't both be 0. As they are equal by give condition, indeed $a = b = 0$ and he minimized expression then becomes $-\frac{cd}{c^2+d^2}$, which has minimum $-\frac{1}{2}$ as $2cd \leq c^2 + d^2$. Moreover, this minimal value is reached by $a = b = 0$ and $c = d > 0$

Solution 2. Consider the Cartesian coordinate system and in it, points $C = (c, 0), D = (0, d)$ and $X = (a, b)$. The line passing through C, D has equation $dx + cy - cd = 0$. From analytic geometry, the formula for distance of point (m, n) to the line $ix + jy + k = 0$ is known. It is $\frac{im+jn+k}{\sqrt{i^2+j^2}}$, where the distance is oriented according to the vertical position of point (m, n) with respect to the given line. With this formula, the distance of point X to the line through C, D is

$$\frac{ad + bc - cd}{\sqrt{c^2 + d^2}}.$$

By rearrranging the given condition on a, b, c, d, we get

$$\left(a-\frac{c}{2}\right)^2+\left(b-\frac{d}{2}\right)^2=\frac{c^2+d^2}{4},$$

which means that the point $X=(a,b)$ lies on the circle with centre $\left(\frac{c}{2},\frac{d}{2}\right)$ and radius $\frac{\sqrt{c^2+d^2}}{2}$, which is exactly the circle with diameter CD. Such point on circle with diameter CD can be at most radius distant from AB, so

$$\frac{ad+bc-cd}{\sqrt{c^2+d^2}}\le\frac{\sqrt{c^2+d^2}}{2}$$

and by rearranging we get the inequality we proved in first solution. Note that the distance from X to CD is nonnegative unless $X=(0,0)$, as X would be above the line CD, because it lies in the first quadrant by the nonnegativity and the halfcircle with diameter CD in first quadrant lies entirely above the line CD. From this, the minimum value must happen for $X=(0,0)$ which gives $-\frac{1}{2}$, as desired.

2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that for every positive integer n

$$a_{n+1}=(n+1)\left(a_n-n+1\right).$$

In terms of a_1, determine the greatest positive integer k such that $\gcd\left(a_i,a_{i+1}\right)=k$ for some positive integer $i\ge 2$. (Note that $\gcd(x,y)$ denotes the greatest common divisor of integers x and y.) (*Patrik Vrba, Slovakia*)

Solution. First, we will prove by induction that $a_n=\left(a_1-1\right)n!+n$ for all $n\geqslant 1$. The base case a_1 is trivial. Now suppose that the closed form holds for some a_n. Then

$$\begin{aligned}a_{n+1}&=(n+1)\left(\left(\left(a_1-1\right)n!+n\right)-n+1\right)\\a_{n+1}&=(n+1)\left(\left(a_1-1\right)n!+1\right)\\a_{n+1}&=\left(a_1-1\right)(n+1)!+(n+1)\end{aligned}$$

Hence, the proof by induction is complete. For the sake of clarity, let $c=a_1-1$. Let p be a prime number dividing $\gcd\left(a_n,a_{n+1}\right)$ for $n\geqslant 2$. Assume $p\leqslant n$. Then $cn!+n\equiv n \bmod p$ however, we have $c(n+1)!+n+1\equiv n+1 \bmod p$ thus we conclude $p>n$. We have $a_{n+1}=(n+1)(cn!+1)$ so then $p\mid(n+1)(cn!+1)$. Assume p divides $cn!+1$. Then $cn!+n\equiv cn!+1 \bmod p$, which implies $n\equiv 1 \bmod p$, which is a contradiction since $p>n$. So either $\gcd(a_n,a_{n+1})=1$ or $p=n+1=\gcd(a_n,a_{n+1})$.

Now suppose $p=n+1$, notice that in this case $p\ge 3$. Then, according to Wilson's theorem $p\mid n!+1$ so we have $p\mid n!+1\mid cn!+c$, which implies $p\mid cn!+c+n-n\Longrightarrow p\mid c-n\Longrightarrow p\mid c+1$, which is however, equal to a_1. The chain of thoughts is reversible so we obtain $\gcd\left(a_{p-1},a_p\right)=p$ if and only if $p\mid a_1$. Therefore, the answer is that k is the biggest odd prime divisor of a_1 or 1 if a_1 is a power of 2.

3. Maryam and Artur play a game on a board, taking turns. At the beginning, the polynomial $XY-1$ is written on the board. Artur is the first to make a move. In each move, the player replaces the polynomial $P(X,Y)$ on the board with one of the following polynomials of their choice:

(a) $X \cdot P(X,Y)$

(b) $Y \cdot P(X,Y)$

(c) $P(X,Y) + a$, where $a \in (-\infty, 2025]$ is an arbitrary integer.

The game stops after both players have made 2025 moves. Let $Q(X,Y)$ be the polynomial on the board after the game ends. Maryam wins if the equation $Q(x,y) = 0$ has a finite and odd number of positive integer solutions (x,y). Prove that Maryam can always win the game, no matter how Artur plays. (*Daniel Holmes, Austria*)

Solution. We claim that Maryam can always achieve that the polynomial on the board at the end of her turn has the form $P(X,Y) = f(XY)$ where $f \in \mathbb{Z}[T]$ can be written as

$$T^n - \sum_{i=0}^{n-1} a_i T^i \qquad \text{for integers } n > 0 \text{ and } a_i \geq 0, \text{ not all of them zero.} \tag{1}$$

As any such f fulfills $\frac{f(x)}{x^n} = 1 - \sum_{i=0}^{n-1} \frac{a_i}{x^{n-i}}$ for all positive real numbers x, which is a strictly increasing function on $(0,\infty)$ with arbitrarily small real values near 0 and tending to 1 for $x \to \infty$, it has exactly one positive real root r. We claim further that Maryam can choose r to be a perfect (integer) square. (Initially, $P(X,Y) = f(XY)$ with $f = T - 1$.) Maryam proceeds as follows:

- If Artur multiplies with X, Maryam multiplies with Y and if Artur multiplies with Y, Maryam multiplies with X. If initially, $P(X,Y) = f(XY)$ was on the board, then the resulting polynomial is $XYf(XY)$, so f changes to $T \cdot f$.

- If Artur adds an integer $0 \leq a \leq 2025$, Maryam adds the number $-a \leq 2025$. The polynomial remains unchanged.

- If Artur adds an integer $a < 0$, write $A(XY)$ for the new polynomial on the board, where $A \in \mathbb{Z}[T]$ is of the form (1). By the discussion above, A has a unique positive real root u. As $A(x) > 0$ for all $x > u$, Maryam can choose an integer $c > u$ (e.g. $c = \lfloor u \rfloor + 1$) and add the negative integer $-A(c^2)$ to the polynomial A on the board. Then the new polynomial on the board has again the form (1) and (by construction) $c^2 \in \mathbb{Z}_{>0}$ as the only positive real root.

Hence, Maryam can always achieve that Q (the polynomial in the end of the game) satisfies $Q = g(XY)$ where g is of the form (1) and has a perfect square s^2, $s \in \mathbb{Z}_{>0}$, as unique positive real root. Now for all pairs of positive integers (x,y), we have $Q(x,y) = 0 \iff g(xy) = 0 \iff xy = s^2$ and it is well known that the number of solutions (x,y) to the last equation is (finite and) odd. (Pairs (x,y) and (y,x) with $x \neq y$ correspond and (s,s) is the only fixed point in this involution, giving an odd number overall.) Hence, Maryam can always win, independent of Artur's moves.