CAPS Match 2025

ISTA, Austria

(First day – 17 June 2025)

1. Let a, b, c, d be nonnegative real numbers for which $a^2 + b^2 = ac + bd$ holds and c, d are not both zero. Find maximum and minimum value of the expression

$$\frac{ad + bc - cd}{c^2 + d^2}.$$

2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that for every positive integer n

$$a_{n+1} = (n+1)\left(a_n - n + 1\right).$$

In terms of a_1, determine the greatest positive integer k such that $\gcd\left(a_i, a_{i+1}\right) = k$ for some positive integer $i \geq 2$. (Note that $\gcd(x, y)$ denotes the greatest common divisor of integers x and y.)

3. Maryam and Artur play a game on a board, taking turns. At the beginning, the polynomial $XY - 1$ is written on the board. Artur is the first to make a move. In each move, the player replaces the polynomial $P(X, Y)$ on the board with one of the following polynomials of their choice:

(a) $X \cdot P(X, Y)$

(b) $Y \cdot P(X, Y)$

(c) $P(X, Y) + a$, where $a \in (-\infty, 2025]$ is an arbitrary integer.

The game stops after each player has made 2025 moves. Let $Q(X, Y)$ be the polynomial on the board after the game ends. Maryam wins if the equation $Q(x, y) = 0$ has a finite and odd number of positive integer solutions (x, y). Prove that Maryam can always win the game, no matter how Artur plays.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Language: English

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4. The plane was divided by vertical and horizontal lines into unit squares. Determine whether it is possible to write integers into cells of this infinite grid so that:

(i) every cell contains exactly one integer

(ii) every integer appears exactly once

(iii) for every two cells A and B sharing exactly one vertex, if they contain integers a and b then at least one of the cells sharing a common side with both A and B contains an integer between a and b.

5. We are given an acute triangle ABC. Point D lies in the halfplane AB containing C and satisfies $DB \perp AB$ and $\angle ADB = 45° + \frac{1}{2}\angle ACB$. Similarly, E lies in the halfplane AC containing B and satisfies $AC \perp EC$ and $\angle AEC = 45° + \frac{1}{2}\angle ABC$. Let F be the reflection of A in the midpoint of arc BAC (containing point A). Prove that points A, D, E, F are concyclic.

6. Find all functions $f\colon (0,\infty) \to [0,\infty)$ such that for all $x, y \in (0,\infty)$ it holds that

$$f(x + yf(x)) = f(x)f(x + y).$$

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