51^{st} Austrian Mathematical Olympiad

National Competition—Final Round (Day 1)

20th June 2020

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1. Let $ABCD$ be a cyclic quadrilateral and let S be the intersection point of its diagonals. Furthermore let P be the circumcenter of the triangle ABS and Q the circumcenter of the triangle BCS. The parallel to AD through P and the parallel to CD through Q intersect in point R.

   Prove that R is on BD.

   (Karl Czakler)

2. There are 2020 points in the plane, some of which are black and the others are green.

   For each black point the following holds: There are exactly two green points that have a distance of 2020 to this black point.

   Determine the smallest possible number of green points.

   (Walther Janous)

3. Let a be a fixed positive integer and (e_n) the sequence defined by $e_0 = 1$ and

   $$e_n = a + \prod_{k=0}^{n-1} e_k$$

   for $n \geq 1$.

   (a) Prove that there are infinitely many primes that divide an element of the sequence.

   (b) Prove that there exists a prime that divides no element of the sequence.

   (Theresia Eisenkölbl)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.

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National Competition—Final Round (Day 2)

27th June 2020

4. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(xf(y)+1) = y + f(f(x)f(y))$$

for all x, $y \in \mathbb{R}$.

(Theresia Eisenkölbl)

5. Let h be a semicircle with diameter AB. An arbitrary point P on the line segment AB is chosen. The line through P that is perpendicular to AB intersects h at point C. The line segment PC divides the area of the semicircle into two parts. In each of them, a circle is inscribed that touches AB, PC and h. The point of tangency of AB and each circle is D and E, respectively, where D is between A and P.

   Prove that the size of the angle $\angle DCE$ does not depend on the choice of P.

(Walther Janous)

6. The players Alfred and Bertrand together determine a polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ of the given degree $n \geq 2$. To do so, in n moves they alternatingly choose the value of one coefficient, where all coefficients must be integers and $a_0 \neq 0$ must hold. Alfred makes the first move. Alfred wins if the final polynomial has an integer root.

   (a) For which n is Alfred able to force a victory if the coefficients a_j are chosen from right to left, that is, for $j = 0, 1, \ldots, n-1$?

   (b) For which n is Alfred able to force a victory if the coefficients a_j are chosen from left to right, that is, for $j = n-1, n-2, \ldots, 0$?

(Theresia Eisenkölbl, Clemens Heuberger)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.