51st Austrian Mathematical Olympiad

National Competition—Preliminary Round—Solutions

21st May 2020

Problem 1. *Let x, y and z be positive real numbers subject to $x \geq y + z$.*

Prove the inequality

$$\frac{x+y}{z} + \frac{y+z}{x} + \frac{z+x}{y} \geq 7.$$

When does equality hold?

(Walther Janous)

Answer. Equality holds if and only if the positive real numbers x, y und z satisfy the condition $x : y : z = 2 : 1 : 1$.

Solution. Evidently, the problem is symmetric with respect to y and z. Therefore, we first focus on these two variables. It is reasonable to guess that the inequality becomes sharper for $y = z$, so we will continue with this in mind.

Since both the inequality and the constraint are homogeneous we can assume without loss of generality that $y + z = 1$.

Therefore, we have to show the inequality

$$x\Big(\frac{1}{y} + \frac{1}{z}\Big) + \frac{1}{x} + \frac{y}{z} + \frac{z}{y} \geq 7$$

for $y + z = 1$ and $x \geq 1$. It is well known that $\frac{y}{z} + \frac{z}{y} \geq 2$ with equality if and only if $y = z = 1/2$. Furthermore, the inequality between the arithmetic and harmonic means yields

$$\frac{1}{y} + \frac{1}{z} \geq \frac{4}{y+z} = 4$$

also with equality if and only if $y = z = 1/2$. Hence, we can sharpen our inequality to

$$4x + \frac{1}{x} \geq 5$$

where $x \geq 1$. Because of $x + 1/x \geq 2$ and $3x \geq 3$ for $x \geq 1$, each with equality exactly for $x = 1$, the proof is complete. The stated equality condition follows immediately from $x = 1$ and $y = z = 1/2$.

(Clemens Heuberger) □

Problem 2. *Let ABC be a right triangle with its right angle in C, and circumcenter U. Points D and E lie on the sides AC and BC, respectively, such that $\angle EUD = 90°$ holds. Furthermore, let F and G denote the feet of D and E on AB, respectively.*

Prove that FG is half as long as AB.

(Walther Janous)

Solution. All angles are considered oriented modulo 180°.

Let U, V and W denote the mid-points of the sides of ABC as shown in Figure 1. It is clear that VW is parallel to AB and half as long, due to the homothety with center C. We wish to show that $FGVW$ is a parallelogram, which will imply $FG = VW$.

It is clear that $VW \parallel FG$ holds, and it therefore remains to show that $GV \parallel FW$, or $\angle GFW = \angle BGV$, holds.

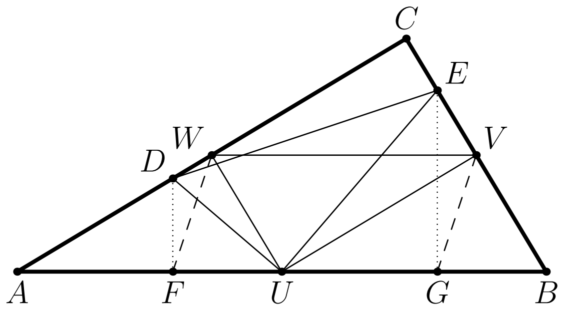

Figure 1: Problem 2

Since the sides of UVW are parallel to the sides of the original right triangle ABC, the quadrilateral $WUVC$ must be a rectangle. Therefore, the desired property is true for $D = W$ (which is equivalent to $E = V$). From now on we will assume $D \neq W$ und $E \neq V$.

Since we have $FU \perp FD$ and $WU \perp WD$, $FUWD$ is an inscribed quadrilateral. Similarly, $DUEC$ is also an inscribed quadrilateral, as we have $UD \perp UE$ and $CD \perp CE$, as is $UGVE$ because of $GU \perp GE$ and $VU \perp VE$. Because of this, we obtain $\angle GFW = \angle UFW = \angle UDW = \angle UDC = 180° - \angle CEU = \angle UEV = 180° - \angle VGU = \angle BGV$, completing the proof.

(Gerhard Kirchner) □

Problem 3. *Three positive integers are written on a blackboard. In each move, the numbers are first assigned the labels a, b and c in a way that $a > \gcd(b, c)$ holds, and then a is replaced with $a - \gcd(b, c)$. The game ends if there is no possible labelling with the desired property.*

Show that the game always ends and always reaches the same three numbers $x \leq y \leq z$ for the same starting numbers.

(Theresia Eisenkölbl)

Solution. Let us first note that in each step the sum of the three numbers will become smaller. Since the sum of three positive integers is a positive integer, the game must end.

Now we will prove that the game always ends with three identical numbers equal to the greatest common divisor of the original three numbers.

We first check that the greatest common divisor of the three numbers is an invariant throughout the game:

$$\gcd(a - \gcd(b, c), b, c) = \gcd(a - \gcd(b, c), \gcd(b, c)) = \gcd(a, \gcd(b, c)) = \gcd(a, b, c).$$

If all numbers are equal, it is obvious that there is no further legal move. It remains to show that there always is a legal move if the three numbers are not all equal.

Therefore, let $a \geq b \geq c$ with $a > c$. Then we have $\gcd(b, c) \leq c < a$ and we can make a move.

We have proved that we always end up with three numbers of the form (n, n, n). Due to the invariance of the greatest common divisor, we have $n = \gcd(n, n, n) = \gcd(a, b, c)$. Therefore, the final result is always (d, d, d) with $d = \gcd(a, b, c)$ as claimed.

(Theresia Eisenkölbl) □

Problem 4. *Determine all positive integers N such that $2^N - 2N$ is the square of an integer.*

(Walther Janous)

Answer. The only solutions are $N = 1$ and $N = 2$.

Solution. We easily check that $N = 1$ and $N = 2$ are solutions. From now on let $N \geq 3$. We show that no further solutions exist.

Obviously, $2^N - 2N$ is even. In order for it to be a perfect square, it must also be divisible by 4, which is the case if and only if N is even.

We therefore set $N = 2k$ with a suitable integer $k \geq 2$. Then $2^{2k} - 4k < 2^{2k}$. The largest even square smaller than $2^{2k} = (2^k)^2$ is $(2^k - 2)^2$, hence we must have

$$2^{2k} - 4k \leq (2^k - 2)^2 = 2^{2k} - 4 \cdot 2^k + 4,$$

which is equivalent to $2^k \leq k + 1$.

But using Bernoulli's inequality we have $2^k = (1 + 1)^k > 1 + 1 \cdot k$ for all $k \geq 2$.

(Walther Janous) □